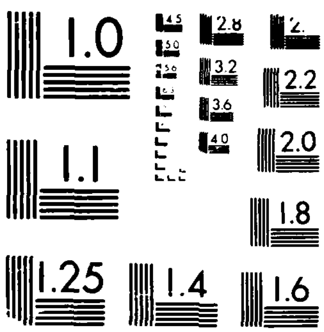
AD-A193 773 AN INTEGRAL EQUATION FOR THE LINEARIZED UNSTEADY SUPERSONIC FLOW OVER A WING(U) DAYTON UNIV OH RESEARCH INST K G GUDERLEY DEC 87 UDR-TR-87-95 AFWAL-TR-87-3107 1/1

UNCLASSIFIED F33615-86-C-3200 F/G 20/4 NL



MICROCOPY RESOLUTION TEST CHART

AFWAL-TR-87-3107

AN INTEGRAL EQUATION FOR THE LINEARIZED UNSTEADY SUPERSONIC FLOW OVER A WING

Karl G. Guderley

University of Dayton
Research Institute
300 College Park
Dayton, Ohio 45469

December 1987

Final Report for the Period May 1986 to August 1987

FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433-6553

NOTICE

This technical report has been reviewed and is approved for publication.

Maxwell Blair for

DR CHARLES KELLER
Project Manager
Aeroelastic Group

Michael H. Shirk

MICHAEL H. SHIRK
Acting Chief
Analysis & Optimization Branch

FOR THE COMMANDER

Henry A Bondaruk Jr

HENRY A. BONDARUK, JR, Colonel, USAF
Chief, Structures Division

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

AD A193773

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS: None
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; Distribution is unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): UDR-TR-87-95	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFWAL-TR-87- 3107
6a. NAME OF PERFORMING ORGANIZATION: University of Dayton Research Institute; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: Flight Dynamics Laboratory (AFWAL/FIBRC) Air Force Wright Aeronautical Laboratories
6c. ADDRESS (City, State and ZIP Code): 300 College Park Dayton, Ohio 45469	7b. ADDRESS (City, State and ZIP Code): Wright-Patterson Air Force Base Ohio 45433-6553
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: Air Force Wright (cont. on back); 8b. OFFICE SYMBOL (If applicable): AFWAL/FIBRC	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F33615-86-C-3200
8c. ADDRESS (City, State and ZIP Code): Wright-Patterson Air Force Base OH 45433-6553	10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2304	N1	22

11. TITLE (Include Security Classification) AN INTEGRAL EQUATION FOR THE LINEARIZED UNSTEADY SUPERSONIC FLOW OVER A WING

12. PERSONAL AUTHOR(S) K. G. GUDERLEY

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Final	FROM May 86 TO Aug 87	1987 December	77

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	Time dependent; linearized potential flow; integral equation; fundamental solutions; unsteady aerodynamics
01	01		
20	04		

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This report derives an integral equation for the linearized supersonic unsteady potential flow over a wing. Every integral equation formulation for a problem that appears originally in the form of a partial differential equation presupposes the availability of a fundamental solution. Such a fundamental solution is available for the problem at hand in the literature. It is rederived here to show its particular properties; further discussions are found in Appendix C. The integral equation originally obtained requires that one carry out a limiting process in which one approaches the planform from above or below. This formulation is brought into a form in which this limiting process no longer appears and one works solely with information available at the planform. Examples which can be treated analytically bring some properties which have a bearing on a numerical approach into sharper focus.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION: Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL: Charles L. Keller	22b. TELEPHONE NUMBER (Include Area Code): 513/255-7384; 22c. OFFICE SYMBOL: AFWAL/FIBRC

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

8a. (cont.)

Aeronautical Laboratories
Air Force Systems Command

PREFACE

The work was performed during the period April 1987 through August 1987 under contract F33615-86-C-3200 entitled "A Study of Linearized Integral Equation for Steady and Oscillatory Supersonic Flow" to the University of Dayton for the Aeroelastic Group, Analysis and Optimization Branch, Structures Division, (AFWAL/FIBRC), Air Force Wright Aeronautical Laboratory under Project No. 2304, Task N1, Work Unit 22 and Program Element 61102F. Dr. Karl G. Guderley of the University of Dayton Research Institute was Principal Investigator. Dr. Charles Keller, AFWAL/FIBRC, was the program manager.

The author would like to express his appreciation for the excellent typing done by the staff of the University of Dayton Research Institute, in particular, Ms. Kathleen Reineke.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (cont.)

SECTION I

INTRODUCTION

In this report the integral equation for the linearized supersonic unsteady potential flow over a wing will be derived and the special properties of its solutions will be discussed by means of limiting cases that can be treated analytically. As in every formulation, by means of an integral equation, the basic prerequisite is a fundamental solution. The expression from which we start here represents a local source with arbitrary time dependence in a system of coordinates fixed with respect to the wing. (The undisturbed air moves with respect to this system of coordinates from left to right.) Only solutions representing outgoing waves are admitted. The fundamental solution used in the integral equation represents a time dependent doublet; it is obtained from the expression for a source by a differentiation into the direction normal to the planform. Use of such a formulation in subsonic flows is shown in References 1 and 2. The corresponding solutions for supersonic flow are given by the same formula. Nevertheless, an ambiguity aries. It seems as if one has two instead of one fundamental solution, each of which satisfies the flow differential equation in the region in which the derivatives exist. At this stage, the typical phenomena of supersonic flow are encountered. One has regions of influence, with a boundary given by the surface of the Mach cone through the point where the source is located. Along this surface the fundamental solutions are singular. Outside the Mach cone they are identically equal to zero. One might try to resolve the above mentioned ambiguity by discussing whether the partial differential equation for the potential is satisfied in the sense of generalized functions as one passes through the surface of the Mach cone. Some ideas which lie in this direction are found in Appendix C. The ambiguity is resolved in Section II.

The point of departure is the fundamental solution for a time dependent local source in a system of coordinates at rest

with respect to the unperturbed air. In this formulation, no ambiguity is encountered. Next one proceeds, still in the same coordinate system, to a distribution of such sources along the x-axis; and subsequently, by a specialization to a localized time dependent source moving along the x-axis from right to left with the desired free-stream velocity. Finally, a transformation to coordinates in which the undisturbed air moves with the desired free stream velocity from left to right is carried out. This gives the desired fundamental solution (a local time dependent source in this last system of coordinates). This procedure is applicable for subsonic as well as supersonic flows. In subsonic flow one finds, of course, the particular solutions used previously. In supersonic flow one finds the sum of two expressions, each of which satisfies the flow differential equation where the derivatives exist. The derivation gives only one linear combination of these expressions and thus removes the ambiguity.

In Section III, familiar formulae expressing the upwash in terms of the motion and/or deformations of the wing are derived.

Section IV formulates the integral equation. The flow field is represented by time dependent doublets distributed over the planform. The source strength in its dependence upon time and location on the planform is unknown at this stage. An essential part in the formulation of the integral equation is a limiting process by which one determines the upwash due to this doublet distribution as one approaches the planform from above or below. (The results are the same.) In the final formulation, this limiting process no longer appears.

In the actual computation, the upwash due to the motions and/or deformations of the wing is built up from expressions each of which is the product of some function of coordinates within the planform and a Hamilton step function in time. It is then sufficient, if one solves the integral equation (separately for each function of the planform coordinates for this special time dependence). For such an upwash, the flow tends towards a steady

state. In contrast to subsonic flows, the steady state is reached within a finite time. The essential physical phenomena occur during this transition time.

Section V deals with two limiting cases that can be treated analytically. The first one, the two-dimensional steady flow, is very simple; the contributions of the integrals in the integral equation are zero. The second example is the one-dimensional unsteady flow. Again, the given upwash is built up by step functions in time. (There is no dependence upon the point of the planform under consideration.) For a step function, one can determine the region in which the integrand is different from zero. Within these regions the iterations can be carried out in a closed form, and one obtains the expected result. For a general upwash (but still with the time dependence given by a step function), the regions of integration are the same, except that there are no doublets upstream of the leading edge. The example shows that the integrands jump from a finite value to zero, as one passes its integrands over the boundary of the region integration. The boundaries of the regions of integration do not coincide with the subdivisions of the planforms imposed by a discretization. In an actual program, one must therefore decide whether to allow these jumps to be smoothed out, or whether it is preferable to identify these boundaries.

In the initial stage of numerical experimentation with a program, examples in which analytic solutions are available are useful. In Section VI an exact solution for the two-dimensional problem with a straight leading edge normal to the flow direction is derived. This is possible because the flow field can then be represented by a source distribution (rather than by a doublet distribution), and because the strength of the sources is directly given by the prescribed upwash. Again, a step function gives the time dependence of the upwash. If the distribution of the upwash over the planform is sufficiently simple, for instance given by polynomials in the coordinates, then the integrals that arise can be evaluated analytically. In testing a

numerical procedure, one will solve the same problem by means of the integral equations for the doublet distribution and examine how well one recovers the analytical results for the potential.

In Appendix A certain integrals are evaluated analytically. Appendix B determines the shape of the regions of integration, if one evaluates the integrals at a distance from the planform. Originally, this was done for the purpose of an extended discussion of the one-dimensional unsteady problem. (The treatment in Section V is restricted to points of the planforms.) The shape of the regions of integration obtained in this Appendix can hardly be foreseen on the basis of the results of Section V. Again, the time dependence is given by a step function. If the time after the step is small, then it is insufficient for the perturbations to propagate the point off the planform under consideration; regions of integration do not exist. If the time is slightly larger than a certain cutoff value, the region of integration is a circle for one of the expressions in the upwash integral; for the other expression there is no region of integration. Regions of integration of this type are not present if one evaluates the upwash at the planform. Beyond a further cutoff point, one finds nonvanishing regions of integration to both terms in the upwash integral. Their boundaries are formed by a hyperbola and parts of a circle. These regions deform steadily into those which one encounters if the upwash is evaluated at the planform.

SECTION II
THE FUNDAMENTAL SOLUTION

The point of departure of this report is a fundamental solution which represents a source located at a fixed point in a system of coordinates with respect to which the air moves with the constant velocity U. The strength of the source depends upon time. In the integral equation from which the flow field is determined, one needs the expression for a doublet. It is obtained from that for a source by a differentiation. In the subsonic case, the expression for the source can be derived directly from the partial differential equation for the potential linearized for small deviations from a parallel flow with this velocity U. It is not entirely obvious how to make the transition to supersonic flows; in fact, in an initial attempt the author was led to an erroneous result. A clear cut answer is obtained if one starts with a particular solution which represents a time-dependent source at a fixed point in a coordinate system which is at rest with respect to the surrounding air, and subsequently derives from it the expression in a system which moves with respect to the air. Similar approaches can probably be found in the literature. The derivation is shown here in order to make this report self-contained.

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be a Cartesian system to coordinates, $\bar{t}$ the time, $\bar{a}$ the velocity of sound in the undisturbed field, L a characteristic length, and $\bar{\phi}(\bar{x},\bar{y},\bar{z},\bar{t})$ the potential. The differential equation for the linearized potential in air at rest is then given by

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{y}\bar{y}} + \phi_{\bar{z}\bar{z}} - \frac{1}{\bar{a}^2}\bar{\phi}_{\bar{t}\bar{t}} = 0 \qquad (1)$$

Introducing dimensionless quantities

$$\tilde{x} = \bar{x}/L, \ \tilde{y} = \bar{y}/L, \ \tilde{z} = z/L, \ \tilde{t} = \bar{a}\bar{t}/L \qquad (2)$$

$$\tilde{\phi}(\tilde{x},\tilde{y},\tilde{z},\tilde{t}) = \bar{a}^{-1}L^{-1}\bar{\phi}(\bar{x},\bar{y},\bar{z},\bar{t})$$

where one expresses $\bar{x},\bar{y},\bar{z}$, and $\bar{t}$ on the right-hand side by $\tilde{x},\tilde{y},\tilde{z}$ and $\tilde{t}$, one obtains

$$\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{y}\tilde{y}} + \tilde{\phi}_{\tilde{z}\tilde{z}} - \tilde{\phi}_{\tilde{t}\tilde{t}} = 0 \tag{3}$$

Hence, for a perturbation with spherical symmetry

$$\tilde{\phi}_{\tilde{r}\tilde{r}} + \frac{2}{\tilde{r}}\,\tilde{\phi}_{\tilde{r}} - \tilde{\phi}_{\tilde{t}\tilde{t}} = 0$$

where

$$\tilde{r} = (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)^{1/2}$$

The particular solution for an outgoing spherical wave is then given by

$$\tilde{\phi} = \frac{1}{\tilde{r}}\, f(\tilde{t} - \tilde{r}) \tag{4}$$

Here f is an arbitrary function of its argument. The potential equation is an expression for conservation of mass. No mass sources arise for $\tilde{r} \neq 0$, for there the partial differential equation is satisfied, but at $\tilde{r} = 0$, mass emerges. This is seen in the following manner. The expression (4) gives a radial velocity

$$\tilde{\phi}_{\tilde{r}} = - \frac{1}{\tilde{r}^2} f(\tilde{t}-\tilde{r}) - \frac{1}{\tilde{r}}\, f'(\tilde{t}-\tilde{r})$$

where f' is the derivative of f with respect to its argument. The mass flow through a surface with radius $\tilde{r}$ is then

$$-4\pi\rho[f(\tilde{t}-\tilde{r}) - \tilde{r}f'(\tilde{t}-\tilde{r})]$$

where ρ is the density. Then for $\tilde{r} = 0$, one has

$$\text{mass flow} = -4\pi\rho f(\tilde{t}).$$

Except for the factor $-4\pi\rho$, which will be unessential in future developments, the strength of the source at time τ is given by $f(\tilde{t})$.

Next, we consider a distribution of such sources along the $\tilde{x}$ axis. One then obtains an axisymmetric flow field. It is sufficient to consider the flow in a meridian plane with coordinates $\tilde{x}$ and $\tilde{y}$ where $\tilde{y}$ is the distance from the axis of symmetry. Later, $\tilde{y}$ will be replaced by $(\tilde{y}^2 + \tilde{z}^2)^{1/2}$.

For a source at the point $\tilde{x} = \tilde{\xi}$, $\tilde{y} = 0$, one has

$$\tilde{r}(\tilde{\xi},\tilde{x},\tilde{y}) = [(\tilde{x}-\tilde{\xi})^2 + \tilde{y}^2]^{1/2} \tag{5}$$

Let the time dependent source strength per unit length of the $\tilde{x}$ axis at the station $\tilde{x}=\tilde{\xi}$ be given by $f(\tilde{t},\tilde{\xi})$ (an unessential factor $-4\pi\rho$ is omitted). Then,

$$\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t}) = \int \frac{f(\tilde{t}-\tilde{r}(\tilde{\xi},\tilde{x},\tilde{y}),\tilde{\xi})}{\tilde{r}(\tilde{\xi},\tilde{x},\tilde{y})}\, d\tilde{\xi} \tag{6}$$

This expression is now specialized to a source moving in the negative x direction with the velocity U. In the $\tilde{x}\tilde{t}$ system, it then moves with the dimensionless velocity $M = U/\bar{a}$. As a first step, we choose

$$f(\tilde{t},\tilde{\xi}) = \tilde{g}(\tilde{\xi})\delta(\tilde{t}-\tau)$$

where $\delta(\tilde{t}-\tau)$ is the Diract delta function. Considering τ as a function of $\tilde{\xi}$, one obtains a source which wanders along the $\tilde{\xi}$ axis and has strength $g(\tilde{\xi})$ at time $\tilde{t} = \tau(\tilde{\xi})$. No mass is expelled at this station for $\tilde{t} \neq \tau(\tilde{\xi})$. The mass expelled in an interval $\tau(\tilde{\xi})-\varepsilon < \tilde{t} < \tau(\tilde{\xi}) + \varepsilon$ with ε arbitrarily small, is $g(\tilde{\xi})$. Now we

choose $\tau(\tilde{\xi})$ so that in a coordinate system, moving to the left with respect to the $\tilde{x},\tilde{y}$ system with the dimensionless velocity M, the source is always at the origin.

Then

$$\tilde{\xi} = -M\tau, \qquad \tau(\tilde{\xi}) = -M^{-1}\tilde{\xi} \tag{7}$$

Substituting this into Eq. (6), one obtains

$$\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t}) = \int_{-\infty}^{+\infty} \frac{\tilde{g}(\tilde{\xi})\delta(\tilde{t}+M^{-1}\tilde{\xi}-\tilde{r}(\tilde{\xi},\tilde{x},\tilde{y}))d\tilde{\xi}}{\tilde{r}(\tilde{\xi},\tilde{x},\tilde{y})} \tag{8}$$

To utilize the defining properties of the delta function, we choose, instead of $\tilde{\xi}$, the argument of the delta function as variable of integration

$$\tilde{t} + M^{-1}\tilde{\xi} - \tilde{r}(\tilde{\xi},\tilde{x},\tilde{y}) = v$$

$$\left[M^{-1} + \frac{\tilde{x}-\tilde{\xi}}{\tilde{r}}\right]d\tilde{\xi} = dv \tag{9}$$

$$d\tilde{\xi}/dv = \frac{M\tilde{r}}{M(\tilde{x}-\tilde{\xi})+\tilde{r}}$$

Eq. (8) now assumes the form

$$\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t}) = \int \frac{M\tilde{g}(\tilde{\xi})\delta(v)dv}{M(\tilde{x}-\tilde{\xi}) + \tilde{r}} \tag{10}$$

here, $\tilde{\xi}$ is regarded as a function of $\tilde{t},\tilde{x},\tilde{y}$, and v. We determine the values of $\tilde{\xi}$ for which the argument of the delta function, namely v, vanishes. Then from the first of Eqs. (9)

$$M\tilde{t} + \tilde{\xi} - M\tilde{r} = 0 \tag{11}$$

It is convenient to introduce

$$\tilde{\xi} + M\tilde{t} = \xi$$

$$\tilde{x} + M\tilde{t} = x \tag{12}$$

$$\tilde{y} = y$$

One then obtains from Eq. (11)

$$\tilde{r} = \xi/M \tag{13}$$

Since r is positive, it follows from the last equation that only such values of $\tilde{\xi}$ are admissible in Eq. (10), for which ξ is positive. After expressing r in terms of ξ, x, and y, one obtains from Eq. (13)

$$\xi - M[(x-\xi)^2 + y^2]^{1/2} = 0.$$

This quadratic equation for ξ has the solutions

$$\xi_i = M(1-M^2)^{-1}[-Mx \pm (x^2 + (1-M^2)y^2)^{1/2}] \; ; \; i = 1,2 \tag{14}$$

One has for $M < 1$

$$|[x^2 + (1-M^2)y^2]^{1/2}| > |x|.$$

The sign of the expression within the brackets in Eq. (14) is therefore determined by the sign of the square root. But, we had observed above that only positive values of ξ give positive values of r. If $M < 1$, only the positive sign of the square root is applicable.

For $M > 1$, one obtains real values of ξ only if the radicand is positive, i.e., for

$$x^2 - (M^2-1)y^2 > 0.$$

The present discussion refers to the points for which the argument of the delta function vanishes, i.e., for which the contributions to the potential, in Eq. (10), is different from zero. The last equation defines a forecone ($x<0$) and an aftercone ($x>0$) of the point $x=0$, $y=0$. As in steady supersonic flows the potential of a source at $x=0$, $y=0$ is different from zero only within the fore- and aftercone.

We write the expression

$$\xi_i = M(M^2-1)^{-1}[Mx \mp (x^2 - (M^2-1)y^2)^{1/2}] \tag{15}$$

For $M > 1$, the sign of the expression within the braces is determined by the sign of Mx. As ξ is positive, only positive values of x are admitted. This restricts the values of x and y to the aftercone of the origin in the xy-system. Both signs of the square root are admissible.

After introducing x and y and expressing $\tilde{r}$ by Eq. (13), the denominator of the integrand in Eq. (10), evaluated for $v = 0$, is found to be

$$M(\tilde{x}-\tilde{\xi}) + \tilde{r} = Mx - M\xi_i + \xi_i/M$$

$$= Mx + (1-M^2)M^{-1}\xi_i \qquad i = 1,2.$$

Substituting ξ_i, Eq. (14), one obtains (both for subsonic and supersonic flow) for $v = 0$,

$$M(\tilde{x}-\tilde{\xi}) + \tilde{r} = \pm [x^2 + (1-M^2)y^2]^{1/2}$$

With this result, one obtains furthermore from Eq. (9)

$$\left.\frac{d\xi}{dv}\right|_{v=0} = \frac{M\tilde{r}}{\pm[x^2 + (1-M^2)y^2]^{1/2}}$$

This shows that at the point where $v = 0$, ξ will increase or decrease, depending upon the sign of the square root.

Now one can evaluate the integral in Eq. (10) on the basis of the definitions of the delta function

$$\int_{-\varepsilon}^{+\varepsilon} \delta(v)dv = 1$$

$$\delta(v) = 0, \quad v \neq 0$$

One obtains

$$\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t}) = \sum_i \frac{\tilde{M}g(\tilde{\xi}_i)}{[x^2 + (1-M^2)y^2]^{1/2}}$$

Here, according to Eq. (12),

$$\tilde{\xi}_i = \xi_i - M\tilde{t}$$

The values of ξ_i are obtained from Eq. (14); only positive values of ξ_i are admitted. We have found that for $M < 1$, only the positive sign of the square root in Eq. (14) is admitted; for $M > 1$, both signs will occur. One thus has for $M < 1$

$$\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t}) = \frac{M\ \tilde{g}(-M\tilde{t}+M(1-M^2)^{-1}(-Mx+(x^2+(1-M^2)y^2)^{1/2})]}{[x^2+(1-M^2)y^2]^{1/2}} \qquad (16a)$$

and for $M > 1$

$$\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t}) = \frac{M}{[x^2 - (M^2-1)y^2]^{1/2}} \qquad (16b)$$

$$\{\tilde{g}(-M\tilde{t} - M(M^2-1)^{-1}[-Mx+(x^2-(M^2-1)y^2)^{1/2}]$$

$$+\tilde{g}(-M\tilde{t} - M(M^2-1)^{-1}[-Mx - (x^2-(M^2-1)y^2)^{1/2}\}$$

for $x > 0$, $|y| < (M^2-1)^{1/2}x$.

So far, the analysis has been carried out in the $\tilde{x}, \tilde{y}, \tilde{t}$ system. This is the system which is at rest in the surrounding air. The variables x and y appearing in the last equation have been defined in Eq. (12); they have been introduced to simplify the calculations. Accordingly, one should express in the above formulae x and y by $\tilde{x}$, and $\tilde{y}$. To obtain the corresponding expression in a system of coordinates in which the source is stationary, one must transform to a system of coordinates which moves with the source, i.e., it moves with the dimensionless velocity M from right to left. Accordingly, one must set

$$\hat{x} = \tilde{x} + M\tilde{t}$$

$$\hat{y} = y$$

$$\phi(\hat{x}, \hat{y}, \hat{t}) = \phi(\tilde{x} - Mt, \tilde{y}, \tilde{t})$$

But $\tilde{x}$ and $\tilde{y}$ are identical with the coordinates x and y introduced in Eq. (12). The right-hand sides of Eqs. (16) have therefore already the desired form if the step from x to $\tilde{x}$ is not carried out. To simplify the expression further, we set

$$M\tilde{g}(-M\xi) = g(\xi)$$

One will remember that we restricted ourselves to one meridian plane with coordinates $\tilde{x}$ and $\tilde{y}$. In general, one must replace $\tilde{y}^2$ by $\tilde{y}^2 + z^2 = y^2 + z^2$. Then one has

for $M < 1$,

$$\phi(x,y,z,t) = \frac{g(t-(1-M^2)^{-1}[-Mx + (x^2+(1-M^2)(y^2+z^2))^{1/2}])}{[x^2 + (1-M^2)(y^2+z^2)]^{1/2}} \tag{17}$$

and for M > 1,

$$\phi(x,y,z,t) = [x^2-(M^2-1)(y^2+z^2)]^{-1/2}$$

$$\{g(t+(M^2-1)^{-1}[-Mx+(x^2-(M^2-1)(y^2+z^2))^{1/2}]$$

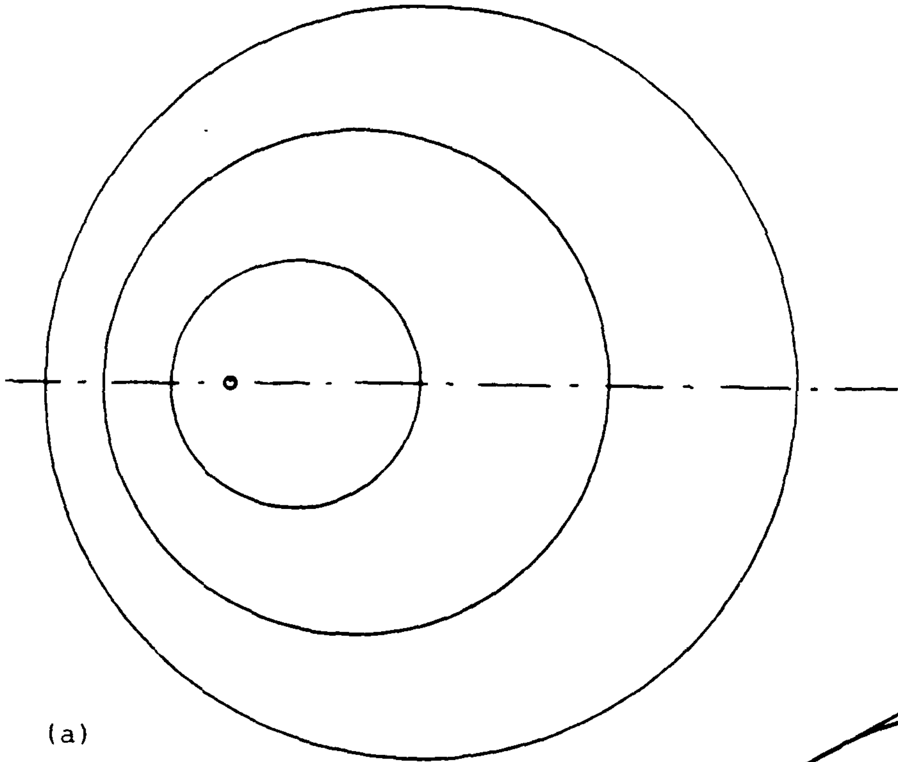
$$+g(t+(M^2-1)^{-1}[-Mx-(x^2-(M^2-1)(y^2+z^2))^{1/2}]\} \qquad (18)$$

for $x > 0$, $y < (M^2-1)^{-1/2}x$, otherwise $\phi=0$.

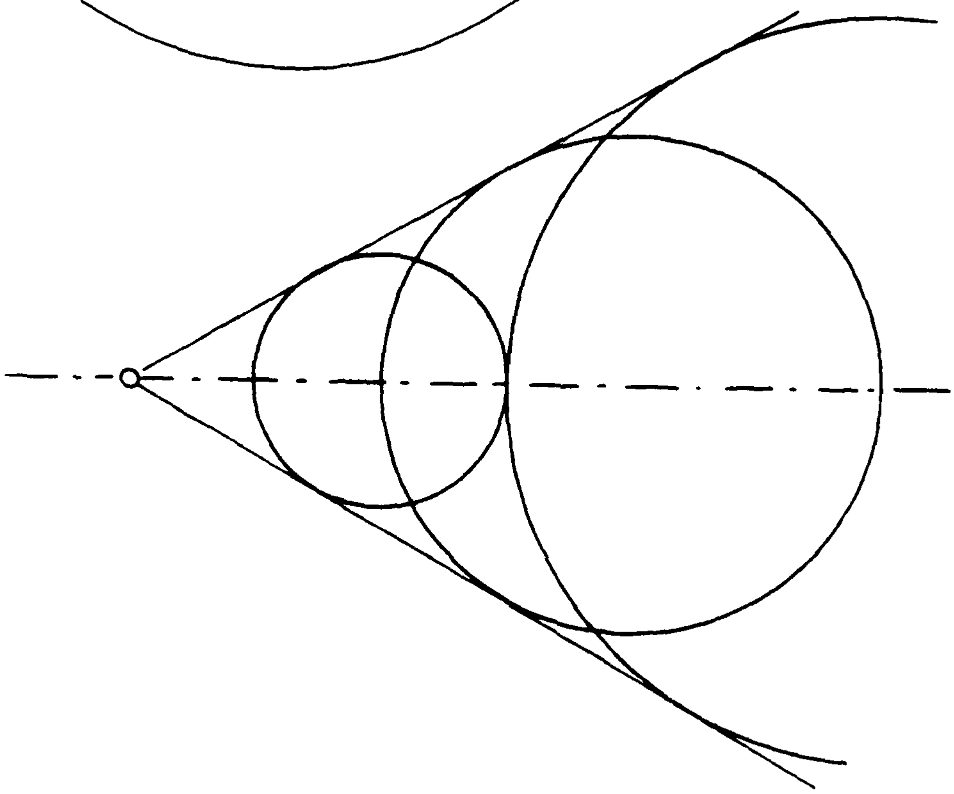
The different appearance of the fundamental solutions in subsonic and supersonic flows finds an explanation in Figures 1a and 1b which show, at a fixed time t, perturbations in a medium at rest caused by momentary emissions of mass from a point moving from right to left with the dimensionless velocity M. It is assumed that at t = 0, the point where the perturbation occurs arrives at the origin of the $\tilde{x},\tilde{y}$ system. For M < 1, the perturbation generated from the individual points lie on circles nested within each other. If the perturbations started at $\tilde{x} = \infty$ at a time $\tilde{t} = -\infty$, then these circles cover the entire $\tilde{x},\tilde{y}$ plane. Only one circle goes through a given point x,y. The center of this particular circle gives the point of the x axis from which the perturbation started.

If M > 1, then all circles lie within a cone through the point $\tilde{x} = -M\tilde{t}, \tilde{y} = 0$. (One will remember that the picture is drawn for a fixed time $\tilde{t}$.) Inside the cone one finds two circles going through each point. More familiar is the interpretation of these figures as perturbation introduced at some point (x,y) of a coordinate system in which the air moves with the dimensionless velocity M from left to right. The perturbations are then found on expanding circles whose center moves with the air particles. (Since these are figures referring to just one instant in time, both interpretations are possible.) All further discussions will be carried out in the x,y,t system.

For an oscillatory source, one has



(a)



(b)

Figure 1. Perturbations at a Fixed Time Generated by a Source Moving from Right to Left with Constant Subsonic Velocity (Fig. 1a) and Constant Supersonic Velocity (Fig. 1b). For Supersonic Flows, the Circles Share an Envelope formed by a Pair of Straight Lines.

$$g(t) = \exp(i\nu t)$$

Then, for $M < 1$,

$$\phi(x,y,z,t) = [x^2 + (1-M^2)(y^2 + z^2]^{-1/2}$$
$$\exp(i\nu(t - M(1-M^2)^{-1}[-Mx + (x^2 + (1-M^2)(y^2+z^2))^{1/2}])) \tag{19}$$

for $M > 1$

$$\phi(x,y,z,t) = 2[x^2 + (M^2-1)(y^2 + z^2]^{-1/2}$$
$$\exp(i\nu(t - M(M^2-1)^{-1}x))\cos(\nu(M^2-1)^{-1}(x^2 - (M^2-1)(y^2+z^2))^{1/2}); \tag{20}$$
$$x > 0, \ |y| < (M^2-1)^{-1/2}x.$$

One notices the occurrence of the cosine instead of an exponential function with imaginary argument. The difference in appearance is the reason why a direct transition from the subsonic fundamental solution to the supersonic fundamental solution is not possible.

The phenomenon has a counterpart in the general unsteady case. As $(x^2 - (M^2-1)(y^2 + z^2))$ tends to zero, (i.e., as one approaches the Mach cone through the point $x = 0$, $y = 0$), the arguments in the two expressions in Eq. (18) approach each other and one can develop g with respect to

$$(M^2-1)(x^2 - (M^2-1)(y^2 + z^2))^{1/2}$$

One obtains

$$\phi(x,y,z,t) = 2[x^2-(M^2-1)(y^2 + z^2)]^{-1/2}[g(t-M(M^2-1)^{-1}x$$

$$+ (1/2(M^2-1)^{-1}g''(t-M(M^2-1)^{-1}x)(M^2-1)^{-2}(x^2-(M^2-1)(y^2+z^2))$$

$$+ (1/24)(M^2-1)^{-1}g^{IV}(t-M(M^2-1)x)$$

$$(M^2-1)^{-4}(x^2-(M^2-1)(y^2+z^2))^2+..$$

This situation can be expressed in the following manner. Let

$$R = [x^2-(M^2-1)(y^2+z^2)]^{-1/2}.$$

At the Mach cone one has $R=0$. Then one has, from Eq. (18)

$$\phi(x,y,z,t)=R^{-1}\{g(t+(M^2-1)^{-1}(-Mx+R))+g(t+(M^2-1)^{-1}(-Mx-R))\}$$

For fixed t and x, the term within the braces is an even function of R.

The original particular solution in the $\tilde{x}\tilde{y}$-system is valid throughout the $\tilde{x}\tilde{y}$-plane, (which is fixed with respect to the undisturbed air) although, because of the introduction of the δ function for f, the individual perturbations are restricted to circles and large regions of the plane remain free of perturbations. Since the final expression in the xy-system (in which the air moves) is obtained merely by a coordinate transformation, the results are valid in the entire xy-plane, in spite of the singularity which occurs at the surface of the Mach cone. This means that the partial differential equation for ϕ is satisfied, even as one passes through this surface.

SECTION III
UPWASH CONDITIONS

Let

$$\hat{x} = xL,\ \hat{y}=yL, \hat{z}=zL, \hat{t}=tL/\bar{a}$$

i.e., $\hat{x}$, $\hat{y}$, $\hat{z}$, and $\hat{t}$ give a system of coordinate corresponding to x,y,z,t, but before these coordinates have been made dimensionless. Let

$$\hat{\phi}(\hat{x},\hat{y},\hat{z},\hat{t}) = \bar{a}L\phi(x,y,z,t) = \bar{a}L\phi(\hat{x}/L,\ \hat{y}/L,\hat{a}/L,a\hat{t}/L)$$

Let the deformation and/or motion of the wing surface be given by

$$\hat{D}(\hat{x},\hat{y},\hat{t}).$$

The upwash condition is then expressed by

$$\hat{\phi}_{\hat{z}} = U\ \frac{\partial\hat{D}}{\partial\hat{x}} + \frac{\partial\hat{D}}{\partial\hat{t}}$$

This is now written in the nondimensional form. Let

$$\hat{D}(\hat{x},\hat{y},\hat{t}) = LD(\hat{x}/L,\hat{y}/L,\bar{a}\hat{t}/L)$$

Then

$$\hat{\phi}_{\hat{z}} = \bar{a}\phi_z,\ \frac{\partial\hat{D}}{\partial\hat{x}} = D_x,\ \frac{\partial\hat{D}}{\partial\hat{t}} = \bar{a}\frac{\partial D}{\partial t}$$

Since U/a = M, one obtains

$$\phi_z = MD_x + D_t \tag{21}$$

SECTION IV

THE INTEGRAL EQUATION FOR TIME-DEPENDENT SUPERSONIC FLOWS

General time dependent supersonic flows over a wing whose planform lies in the xy-plane can be represented by doublets (oriented in the z-direction) distributed over the planform and, if one has a subsonic trailing edge, over part of the wake. Fundamental solutions for such doublets are obtained from those for sources by a differentiation with respect to z. (In the present context a normalization of the sources or doublets is not needed.) The evaluation of the upwash due to this (unknown) doublet distribution requires a further derivative with respect to z. The desired integral equation for the doublet distribution is obtained by equating the upwash so found, with the upwash prescribed by the upwash condition Eq. (21).

Let $\xi\eta$ be a system of Cartesian coordinates in the plane of the planform; ξ corresponds to x and η to y. Let the doublet strength at a point of the planform (ξ,η) be given by $h(t,\xi,\eta)$. Let furthermore

$$\beta = (M^2 - 1)^{1/2} \tag{22}$$

To obtain the potential due to a source at the point $\xi=x$, $\eta=y$, one must replace on the right hand side of Eq. (18) x by $(x-\xi)$ and y by $(\eta-y)$.

We introduce

$$R = [(x-\xi)^2 - \beta^2 z^2 - \beta^2(\eta-y)^2]^{1/2} \tag{23}$$

and

$$ret_{1,2} = \beta^{-2}[M(x-\xi) \pm R] \tag{24}$$

here the upper and lower sign pertain to the former indices $i=1$ and $i=2$.

The symbol ret stands for retardation. Roughly speaking, it gives the time required for a perturbation to propagate from a point ξ,η to a point x,y,z.

In the coming discussions x,y and t are kept constant. In some of the functions that will occur, these quantities are not listed as arguments.

The potential due to a source distribution in supersonic flow is then given by

$$\phi^{(s)}(t,x,y,z) = \iint (1/R)[h(t-\mathrm{ret}_1(x,y,\xi,\eta),\xi,\eta) + h(t-\mathrm{ret}_2(x,y,\xi,\eta),\xi,\eta)]\,d\xi d\eta \tag{25}$$

The region of integration comprises all points of the planform and sometimes of the wake, that lie within the forecone of the point x,y,z, for only then is R real and $x-\xi > 0$. The boundary of the region consists of the hyperbola $R=0$, Eq. (23) and of the leading edge. The vertex of the hyperbola lies at

$$(x-\xi) = \beta z, \quad \eta=y \tag{26}$$

for $z > 0$

To bring the portion of the boundary formed by this hyperbola into a simple form we introduce

$$\eta = y + \beta^{-1} q\bar{\eta} \tag{27}$$

with
$$q(x-\xi,z,\beta) = [(x-\xi)^2-\beta^2 z^2]^{1/2} \tag{27a}$$

One notes
$$\frac{\partial q}{\partial z} = \frac{-\beta^2 z}{q} \tag{28}$$

Then
$$R = q(1-\bar{\eta}^2)^{1/2} \tag{29}$$

$$d\eta = \beta^{-1} q \, d\bar{\eta} \tag{30}$$

The hyperbola R=0 is then transformed into three straight lines in the $\xi\bar{\eta}$ plane

$$\bar{\eta} = \pm 1$$

and $$q = 0$$

Only the line q = 0 depends upon z. The condition q=0 becomes

$$x-\xi = \beta z$$

Let ξ_0 be the smallest value of ξ which occurs within the region of integration. Eq. (25) can be written in the form

$$\phi^s(t,x,y) = \int_{\xi o}^{x-\beta z} \left(J_1(\xi_1 z) + J_2(\xi,z)\right) d\xi \tag{31}$$

where after substitution of the retardation

$$J_{1,2}(\xi,z) =$$

$$\beta^{-1} \int_{\bar{\eta}_{lower}(\xi)}^{\bar{\eta}_{upper}(\xi)} (1-\bar{\eta}^2)^{-1/2} h(t-\beta^{-2}[M(x-\xi) \pm q(1-\bar{\eta}^2)^{1/2}], \xi, y+\bar{\beta}^1 q\bar{\eta}) d\bar{\eta} \tag{32}$$

For simplicity x,y, and t are not shown as arguments of J_1 and J_2. For those values of ξ for which the upper and lower limits of $\bar{\eta}$ are formed by the hyperbola R=0 these limits are ± 1. For values of ξ where one or both limits for $\bar{\eta}$ are formed by the

leading edge the limits depend upon ξ and z. At those limits the potential and therefore also the function h is zero.

To obtain the potential of a doublet distribution, with the doublets oriented in the z direction, one must differentiate this expression with respect to z. We consider first a region $\xi_1<\xi<x$ where ξ_1 is sufficiently close to x so that the limits of $\bar{\eta}$ are $\pm$ 1. Let $\phi^{(s,I)}$ and $\phi^{(d,I)}$ be the contributions to the potentials $\phi^{(s)}$ and $\phi^{(d)}$ from this region. The point of departure is

$$\phi^{(s,I)} = \int_{\xi_1}^{x-\beta z} (J_1(\xi,z) + J_2(\xi,z))d\xi \tag{33}$$

In differentiating this expression with respect to z one obtains one term denoted by $\phi^{(dIa)}$ due to the differentiation with respect to the upper limit and a second term, denoted by $\phi^{(dIb)}$ due to the differentiation of the integrand.

One finds

$$\phi^{(dIa)} = -\beta[J_1(\xi_1,z) + J_2(\xi_1,z)]\Big|_{\xi=x-\beta z}$$

According to Eq. (27a) one has q = 0 for ξ = x-βz; the function h in Eq. (32) is no longer dependent upon $\bar{\eta}$. Now

$$\int_{-1}^{+1} (1-\bar{\eta}^2)^{-1/2}d\bar{\eta} = \pi$$

Therefore

$$\phi^{(dIa)} = -2\pi h(t-M\beta^{-1}z,\ x-\beta z,y) \tag{34}$$

The dependence of the functions J_1 and J_2 in Eq. (33) upon z enters through the function q (see Eq. 32). As q appears in two

of the arguments of the function h one obtains from each of the functions J_1 and J_2 two terms for their derivatives with respect to z (superscripts b and c). We denote derivatives of h with respect to its first second or third argument respectively by $h^{(1)}$, $h^{(2)}$, and $h^{(3)}$. Then with Eq. (28)

$$(\partial J_1^{(b)}/\partial z) + (\partial J_2^{(b)}/\partial z) =$$

$$= z\beta^{-1}q^{-1}\int_{-1}^{+1}\{h^{(1)}(t-\beta^{-2}[M(x-\xi)+q(1-\bar{\eta}^2)^{1/2}],\xi,y+\beta^{-1}q\bar{\eta})$$

$$-h^{(1)}(t-\beta^{-2}[M(x-\xi)-q(1-\bar{\eta}^2)^{1/2}],\xi,y+\beta^{-1}q\bar{\eta})\}d\bar{\eta} \qquad (35)$$

and

$$(\partial J_1^{(c)}/\partial z + (\partial J_2^{(c)}/\partial z)$$

$$=-zq^{-1}\int_{-1}^{+1}\bar{\eta}(1-\bar{\eta}^2)^{-1/2}\{h^{(3)}(t-\beta^{-2}[M(x-\xi)+q(1-\bar{\eta}^2)^{1/2}],\xi,y+\beta^{-1}q\bar{\eta})$$

$$+h^{(3)}(t-\beta^{-2}[M(x-\xi)-q(1-\bar{\eta}^2)^{1/2}],\xi,y+\beta^{-1}q\bar{\eta})\}d\bar{\eta} \qquad (36)$$

At the upper limit of the integral (31), one has q=0. The factor q^{-1} in the last Eqs. (35) and (36) might cause the integrals in Eq. (31) to diverge, but because of the special form of these equations this does not happen; the reasons are different in the two cases. In Eq. (35) the two terms of the integrand agree for q=0, except for the signs. Accordingly, the integrand is O(q); observing that for q=0,

$$x-\xi=\beta z$$

and

$$\int_{-1}^{+1} (1-\bar{\eta}^2)^{1/2} d\bar{\eta} = \pi/2$$

one finds

$$\lim_{q\to 0} \left((\partial J_1^{(b)}/\partial z)+(\partial J_2^{(b)}/\partial z)\right) = -z\pi\beta^{-3}h^{(1,1)}(t-\beta^{-1}Mz, x-\beta z, y)$$

Regarding Eq. (36) one observes that

$$\int_{-1}^{+1} \bar{\eta}(1-\bar{\eta}^2)^{-1/2}h^{(3)}(t-\beta^{-2}[M(x-\xi)\pm q(1-\bar{\eta}^2)^{1/2}], \xi, y)d\bar{\eta}=0 \qquad (37)$$

because of the antisymmetry of the integrand with respect to $\bar{\eta}$. Notice that only the last argument of $h^{(3)}$ differs in Eq. (36) and (37). Subtracting the two forms of Eq. (37) from Eq. (36) developing the expressions with respect to the last argument and observing that

$$\int_{-1}^{+1} \frac{\bar{\eta}^2 d\bar{\eta}}{(1-\bar{\eta}^2)^{1/2}} = \frac{\pi}{2}$$

one obtains

$$\lim_{q\to 0} \left((\partial J_1^{(c)}/\partial z)+(\partial J_2^{(c)}/\partial z)\right)=-z\pi\beta^{-1}h^{(3,3)}(t-\beta^{-1}Mz, x-\beta z, y)$$

Again this expression is bounded. This procedure requires that one carry out the integration with respect to $\bar{\eta}$ first. One can now carry out the limiting process $z\to 0$. Since the z derivatives of J_1 and J_2 contain a factor z, their contributions vanish in this limit. One therefore obtains

$$\lim_{z\to 0} \phi^{(dI)} = \lim_{z\to 0} \phi^{(dIa)} = -2\pi h(t,x,y) \qquad (38)$$

Accordingly, the function h represents, except for the factor -2π, the potential at the upper side of the wing.

In order to obtain the upwash due to the expression $\phi^{(dI)}$ one must form the derivative with respect to z and subsequently perform the limiting process $z \to 0$. One obtains from Eq. (34)

$$\lim_{q\to 0} \phi_z^{(dIa)} = 2\pi\left(M\beta^{-1}h^{(1)}(t,x,y) + \beta h^{(2)}(t,x,y)\right)$$

or in more conventional notation

$$\lim_{z\to 0} \phi_z^{(dIa)} = 2\pi\left(M\beta^{-1}h_t(t,x,y) + \beta h_x(t,x,y)\right)$$

Eqs. (35) and (36) have the form zF(z). Then

$$\lim_{z\to 0} \frac{d}{dz}\,(zF(z)) = F(0)$$

Taking into account that $q\big|_{z=0} = x - \xi$ one obtains

$$\begin{aligned}\lim_{z\to 0} \phi_z^{(dIb)} = \int_{\xi_1}^{x} \frac{d\xi}{x-\xi}\Big(\int_{-1}^{+1} \Big\{ \beta^{-1}\big[& h^{(1)}(t-\beta^{-2}[M(x-\xi)+(x-\xi)(1-\bar{\eta}^2)^{1/2}],\xi,y+\beta^{-1}(x-\xi)\bar{\eta}) \\ & -h^{(1)}(t-\beta^{-2}[M(x-\xi)-(x-\xi)(1-\bar{\eta}^2)^{1/2}],\xi,y+\beta^{-1}(x-\xi)\bar{\eta})\big] \\ & -\bar{\eta}(1-\bar{\eta}^2)^{-1/2}\big[h^{(3)}(t-\beta^{-2}[M(x-\xi)-(x-\xi)(1-\bar{\eta}^2)^{1/2}],\xi,y+\beta^{-1}(x-\xi)\bar{\eta}) \\ & \qquad +h^{(3)}(t-\beta^{-2}[M(x-\xi)-(x-\xi)(1-\bar{\eta}^2)^{1/2}],\xi,y+\beta^{-1}(x-\xi)\bar{\eta})\big]\Big\}\,d\bar{\eta}\end{aligned}$$

In the contribution of $\phi^{(sII)}$ $\phi^{(s)}$ one has $q \neq 0$. The upper limit of ξ, namely ξ_1, is fixed. In differentiating J_1 and J_2 with respect to z one obtains, in principle, contributions from the z-derivatives of the limits ξ_{lower} and ξ_{upper}. While these limits are fixed in the $\xi\eta$-system they will depend upon z in the $\xi\bar{\eta}$-system. These limits are given by the leading edges and there

$h=0$. At subsonic leading edges a square root singularity appears. There might be a question whether or not this has an effect on the second derivative with respect to z. This has been discussed in considerable detail for the steady case (Ref. 4) with the result that in the limit $z \to 0$ there are no contributions due to these limits. One therefore finds

$$\lim_{z \to 0} \phi_z^{(sII)} = 0$$

Moreover, $\lim_{z \to 0} \phi_z^{(dII)}$ will have the same integrand as $\phi_z^{(dI)}$; only the region of integration is different Returning to the original coordinate η one obtains the following expression.

Let

$$R = [(x-\xi)^2 - \beta^2(\eta-y)^2]^{1/2} \tag{39}$$

$$ret_{1,2} = \beta^{-2}[M(x-\xi) \pm R]$$

Then

$$\phi_z^d(t,x,y,z=0) = 2\pi(M\beta^{-1}h_t(t,x,y) + \beta h_x(t,x,y))$$
$$+\iint \{h^{(1)}(t-ret_1,\xi,\eta)-h^{(1)}(t-ret_2,\xi,\eta)$$
$$-\beta^2((\eta-y)/R)[h^{(3)}(t-ret_1,\xi,\eta)+h^{(3)}(t-ret_2,\xi,\eta)]\}\frac{d\xi d\eta}{(x-\xi)^2} \tag{40}$$

The region of integration has as boundaries the leading edge and the two Mach waves through the point (x,y) into which the original hyperbola degenerates for $z \to 0$. Equating this expression with the upwash found in Eq. (21) one obtains the desired integral equation for the $\eta(t,x,y)$. From Eq. (38) one obtains the potential and hence all quantities of physical interest. The integrands in Eq. (40) occur with a retarded time argument. In essence they are known from the results for preceding time steps.

If data up to a certain time are known, then Eq. (40) allows one to evaluate $h_t(t,x,y)$. This is the essential step in an integration procedure.

We add the following observation. In most of the practical problems the upwash computed from the displacement and or deformation of the wing Eq. (21) can be written in the form

$$w(t,x,y) = \sum_k c_k(t)\, w_k(x,y)$$

where the number of functions $w_k(x,y)$ is rather small. These functions are considered as known. Whether or not the functions $c_k(t)$ are known in advance depends upon the nature of the problem. In aeroelastic problems the determination of the aerodynamic response is done in a preparatory phase and then the deformation are not yet known. In the aeroelastic equations the $c_k(t)$ will then appear as dependent variables. It is practical to express the functions $c_k(t)$ in the form

$$c_k(t) = c_k^{(0)} + \int \dot{c}_k(\tau)H(t-\tau)d\tau \qquad (41)$$

Here $\dot{c}_k$ denotes the derivative of c_k with respect to its argument and $H(\hat{t})$ is the Hamilton step function

$$H(\hat{t}) = 1 \;, \text{ for } \hat{t} > 0$$

$$H(\hat{t}) = 0 \;, \text{ for } \hat{t} < 0$$

One observes that $H(t-\tau) = 1$ in Eq. (41) because $t>\tau$. We assume that no perturbations are present at $t = 0$.

Accordingly, it suffices if one solves the integral equations for

$$w(t,x,y) = w_k(x,y)\; H(t-\tau)$$

separately for all functions $w_k(x,y)$ that are of interest. Since t does not occur explicitly in the formulation of the problem, it is even sufficient if one solves it for

$$w(t,x,y) = w_k(x,y)H(t).$$

All other solutions can be built up from results obtained with these expressions for the upwash. Since for $t>0$, $H(t)$ is independent of t, the solution for $h(t,x,y)$ represents the transition to a steady state with the boundary condition $w = w_k(x,y)$. This happens within finite time.

To become familiar with the special properties of the integral equation, we consider special cases in the next sections.

SECTION V

SPECIAL CASES

A nearly trivial example is obtained for a two dimensional steady flow in the xz-plane. Then $w = w(x)$, $w_t = 0$, $h^{(1)} = h_t = 0$ $h^{(3)} = h_y = 0$. The integrals in Eq. (40) vanish and one obtains

$$w = w(x) = 2\pi\beta h_x t$$

One has, according to Eq. (38)

$$\phi = -2\pi h(x)$$

By definition $\qquad w = \phi_z.$

Therefore, from Eq. (40)

$$\phi_z = -\beta\phi_x, \text{ valid for } z=0$$

This relation is a solution of the equation for two-dimensional steady flow

$$-\beta^2\phi_{xx} + \phi_{zz} = 0.$$

More insight is obtained by the one-dimensional unsteady case. Here all solutions can be built up from the solution of the problem with

$$w = w(t) = H(t)$$

From the partial differential equation, which in this case simplifies to

$$\phi_{zz} - \phi_{tt} = 0,$$

one expects to find

$$\phi_t = -\phi_z \tag{42}$$

We therefore set, tentatively,

$$h_t = cH(t) \;,\; h_x = 0,\; h_y = 0 \tag{43}$$

where the constant c remains to be determined. The integral equation then gives

$$\phi_z = H(t) = c[2\pi M\beta^{-1}H(t) + I_1 - I_2] \tag{44}$$

where

$$I_{1,2} = \iint \frac{H(t-\beta^{-2}(M(x-\xi)\pm R)}{(x-\xi)^2}\, d\xi d\eta \tag{44a}$$

$$R = [(x-\xi)^2 - \beta^2(\eta-y)^2]^{1/2}$$

We immediately set x=0 and y=0 (since all points of the xy-plane are equivalent) and consider a fixed time t. The region of integration lies between the Mach waves through the point

$$\xi = x = 0 \quad \eta = y = 0$$

i.e., between the straight lines

$$|\eta| = \beta^{-1}\xi, \quad \xi<0$$

As a further boundary one has the curve for which the arguments of the functions H vanish

$$t-\beta^{-2}(-M\xi\pm R) = 0 \tag{45}$$

The more general case, in which R is given by

$$R = [(x-\xi)^2 - \beta^2 z^2 - \beta^2 (\eta-y)^2]^{1/2}$$

is discussed in Appendix B. One finds for both signs of R in Eq. (44a)

$$(\xi-(x-Mt))^2 + (\eta-y)^2 = t^2 - z^2$$

This is a circle with center at $\xi = x-Mt$, $\eta=y$ and radius $(t^2-z^2)^{1/2}$.

In the present case z=0, x=0, y=0. The last equation reduces to

$$(\xi+Mt)^2 + \eta^2 = t^2.$$

The areas of integration for the integrals I_1 and I_2 are then given respectively by Figures 2a and 2b. Since in the areas of integration H=1, one obtains

$$I_1 - I_2 = -\iint \frac{d\xi d\eta}{\xi^2}$$

where the area of integration is the circle

$$(\xi+Mt)^2 + \eta^2 = t^2$$

Setting

$$\xi + Mt = \hat{\xi}t$$

$$\eta = \hat{\eta}t$$

one obtains

$$I_1-I_2 = -\iint \frac{d\hat{\xi}d\hat{\eta}}{(\hat{\xi}-M)^2} \quad , \hat{\xi}^2 + \hat{\eta}^2 < 1$$

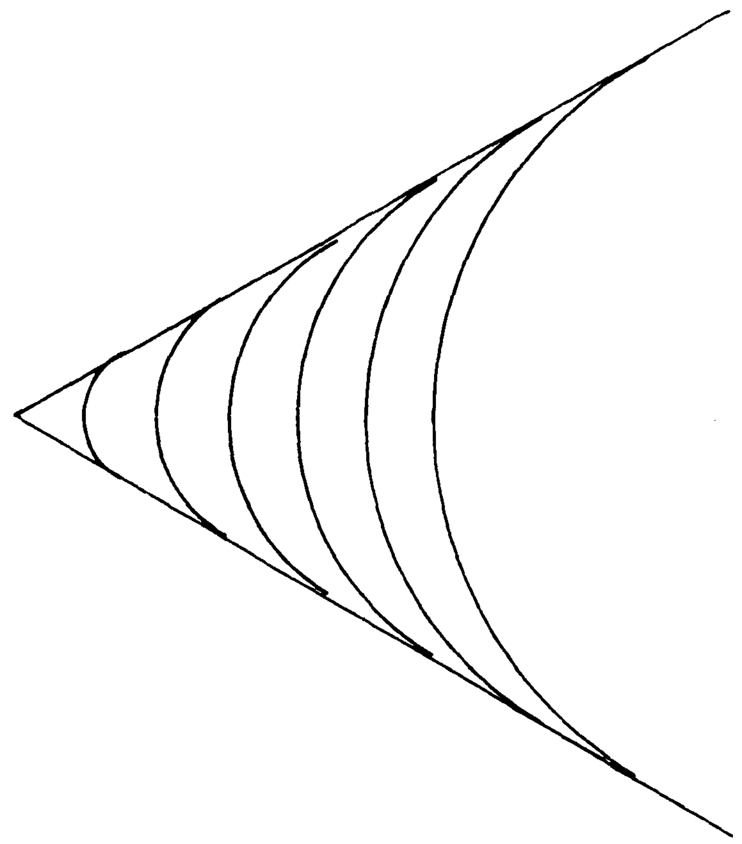


Figure 2a. Regions of Integration for z=0 at Different Times. The Boundaries are Formed by the Mach Lines and by Parts of Circles.

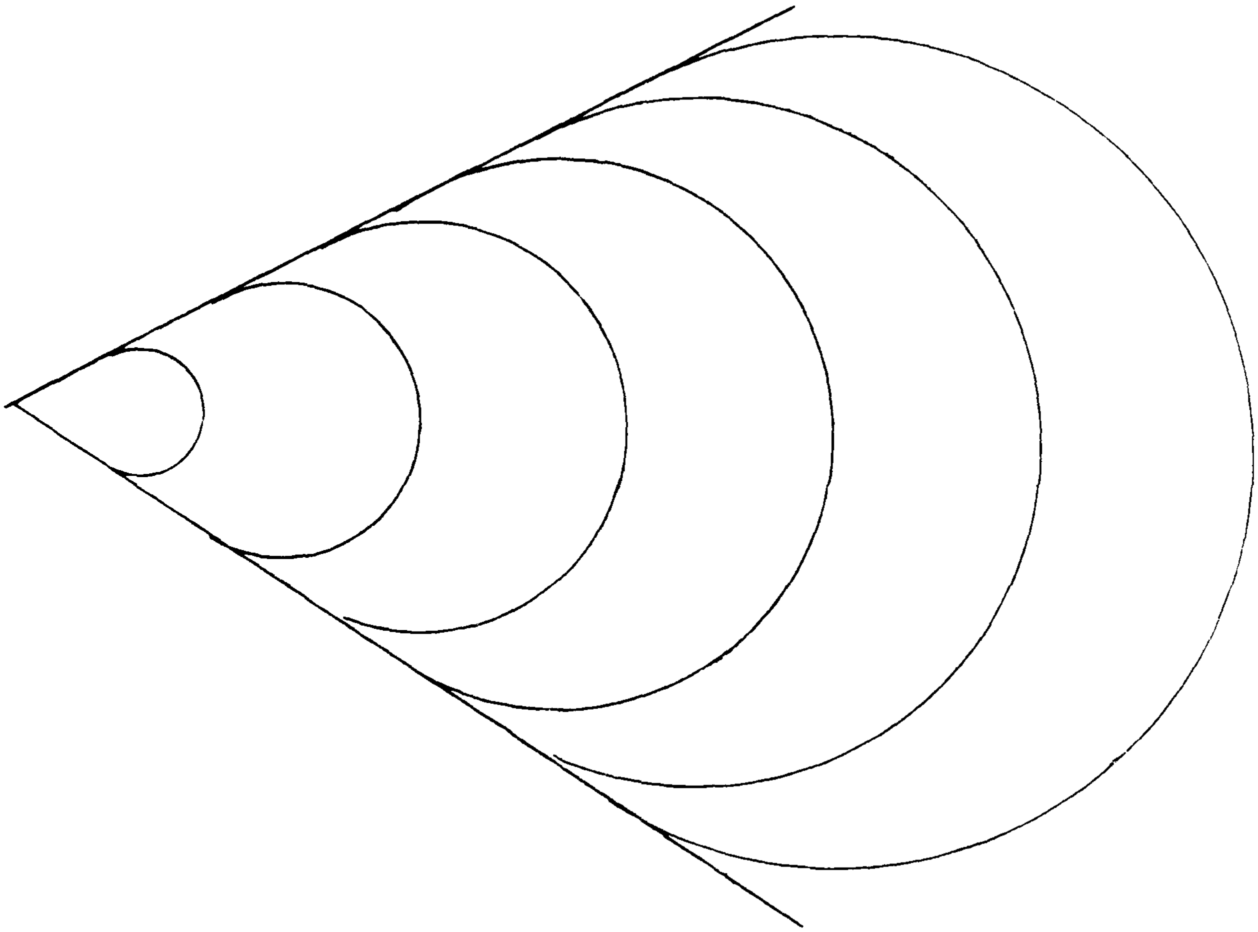


Figure 2b. Regions of Integration for z=0 at Different Times. The Boundaries are Formed by the Mach Lines and Parts of Circles.

First integrating with respect to $\hat{\eta}$ one obtains

$$I_1 - I_2 = -2\int_{-1}^{+1} \frac{d\hat{\xi}(1-\hat{\xi}^2)^{1/2}}{(\hat{\xi}-M)^2}$$

The integral can be evaluated in terms of elementary functions (Appendix A).

$$I_1 - I_2 = 2\pi(1-(M/\beta))$$

Substituting this into Eq. (44) one obtains

$$H(t) = 1 = c\ 2\pi$$

Hence $c = (2\pi)^{-1}$

With Eq. (43) one finds

$$h_t = (2\pi^{-1})\ H(t)$$

and therefore

$$\phi_t = -H(t)$$

This is indeed the result anticipated in Eq. (42)

The jump of ϕ_t at $t=0$ encountered here is typical for problems whose time dependence is given by a step function. This is no reason to abandon such an approach; it is the key step to the solution for general time dependence. It may, however, make special measure in a numerical approach necessary. ϕ and its derivatives are functions of t, x and y. In the numerical work one will therefore generate tables which, for individual points (x,y) of the planform, give ϕ (and probably also ϕ_t) as functions of t. ϕ_t and ϕ_η appear in the integral equation with the first

argument t-ret, where the retardation depends on $x-\xi$ and $\eta-y$. In general an interpolation will be necessary, to find the quantities in question for these arguments.

As far as the determination of the integrands at given points (ξ,η) is concerned, this does not generate a difficulty. One will always have $t=0$ as first value of the independent variable in the above mentioned tables for the selected points (x,y). If the argument t-ret < 0, one always obtains zero; for t-ret > 0, one will use interpolation.

In this example, we have seen that the vanishing of ϕ_t for $\text{t-ret}_1 < 0_1$ defines a boundary for the region of integration. Describing the situation in a different manner, we consider the integrand versus η at a fixed value of ξ, Figure 3. At t-ret = 0 the integrand jumps from zero to some finite value. The jump will in general not occur at one of the chosen points (x,y). If one uses interpolation then one replaces the jump by a ramp. The error so introduced will be small, if the points (ξ,η) are closely spaced. It may, however, be preferable, to identify in advance the point where the jump occurs (here the value of η) and modify the integration formula accordingly.

Some numerical experiments will be needed to arrive at a satisfactory compromise between complexity of the programs, computational labor, and accuracy.

In the next section we treat by analytical means the two-dimensional problem with a supersonic leading edge and $w(x,y) = H(t)$. In this case one can represent the solution by a distribution of known sources and find the potential by a direct integration. The result can serve to test the accuracy of numerical approaches where the same problem is solved by means of the integral equation (which uses the concept of a doublet distribution).

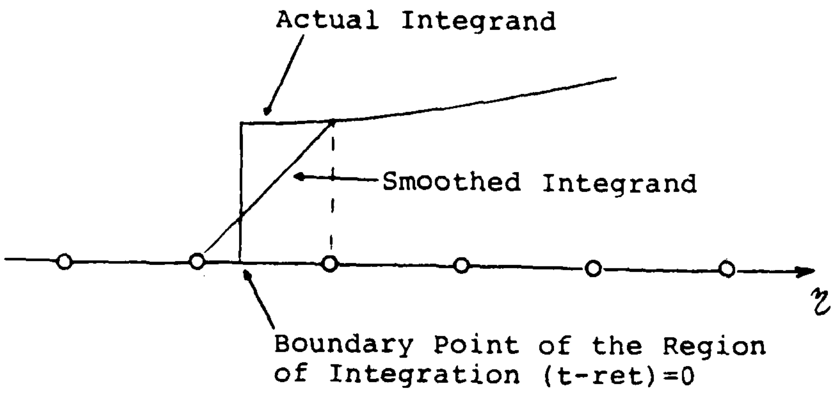


Figure 3. Integrand versus η at a Fixed Value of ξ. The Circles are Points at which the Integrand is Tabulated if the Problem is Discretized. If the Boundary Point of the Region of Integration is not Identified then the Jump at the Boundary will be Smoothed out over one mesh.

SECTION VI

THE TWO-DIMENSIONAL PROBLEM WITH A STRAIGHT SUPERSONIC LEADING EDGE

The problem can always be reduced to one in which the leading edge is normal to the flow direction, by considering only the velocity component normal to the leading edge.

If the potential is represented by sources with strength $h(t,\xi,\eta)$, then according to Eq. (38), taking into account that

$$\phi_z^s(t,x,y) = \phi^d(t,x,y) = -2\pi h(t,x,y)$$

We choose

$$w(t,x,y) = 2\pi H(t)$$

then

$$h(t,\xi,\eta) = -H(t)$$

and from Eq. (25)

$$\phi^{(s)}(t,x) = -\iint 1/R[H(t-\beta^{-2}[M(x-\xi)+R])+H(t-\beta^{-2}[M(x-\xi)-R]d\xi d\eta$$

$$R = [(x-\xi)^2 - \beta^2\eta^2]^{1/2}$$

The areas of integration can be reduced to those points of the plan form for which the arguments of the unit step function H are positive. Accordingly one writes

$$\phi^{(s)}(t,x) = -[\iint_{R_1} \frac{d\xi d\eta}{R} + \iint_{R_2} \frac{d\xi d\eta}{R}] \qquad (46)$$

where R_1 and R_2 are the respective areas. The regions R_1 and R_2 are bounded by the Mach waves through the point $\xi=x$, $\eta=0$, i.e.,

$$\eta = \pm \beta^{-1}(x-\xi) \tag{47}$$

and portions of the circle

$$(x-\xi - Mt)^2 + \eta^2 = t^2 \tag{48}$$

This circle is tangent to the Mach waves. Tangency occurs at $x-\xi = \frac{M^2-1}{M} t$. These points of tangency divide the circle into two parts; they constitute portions of the boundaries of R_1, and R_2. Furthermore the regions R_1 and R_2 are restricted to the planform.

We call temporarily basic regions of integration the regions with the boundaries given by Eq. (47) and (48). Their shape is seen in Figure 4, case a. The actual regions of integration R_1 and R_2 differ from these basic regions on account of the fact, that upstream of the leading edge there is no upwash and therefore no sources are encountered. For fixed t the shape and size of the basic regions of integration is always the same. If x is varied then the entire figures shift in the x direction. For different values of x different parts of the basic regions of integration are cut off by the leading edge. Thus one obtains different cases depending upon the value of t/x. They are shown in Figures 4. In determining which configuration applies in a specific case one remembers that the most upstream point of the circle lies at $\xi=x-(M+1)t$, the point of tangency between the Mach wave and the circle lies at $\xi=x-M^{-1}(M^2-1)t$, and the most downstream point of the circle lies at $\xi=x-(M-1)t_2$. The leading edge lies at $\xi=0$. One thus obtains the following cases, Figures 4.

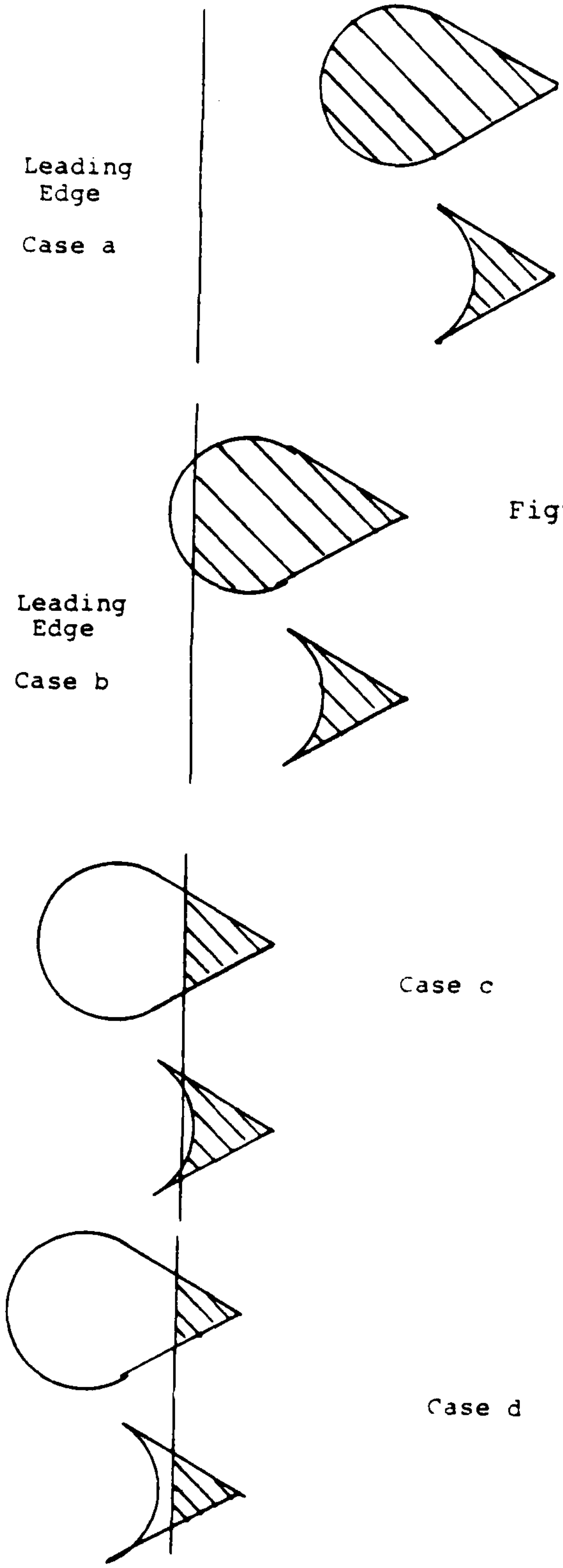


Figure 4. Boundaries of Integration for Fixed z and Different Values of x in a Supersonic Flow with Straight Leading Edge Normal to the Flow Direction.

For each Value of x one has two Different Areas of Integrations. The Shape of the Areas of Integration is the same for all Values of x, except that Parts Upstream of the Leading Edge must be Excluded.

Case a $\quad t/x < (M+1)^{-1}$

Case b $\quad (M+1)^{-1} < t/x < M(M^2-1)^{-1}$

Case c $\quad M(M^2-1)^{-1} < t/x < (M-1)^{-1}$

Case d $\quad (M-1)^{-1} < t/x$

We introduce

$$\hat{\eta} = \frac{\eta\beta}{x-\xi}, \text{ then } \eta = \beta^{-1}\hat{\eta}(x-\xi) \tag{49}$$

and

$$R = (x-\xi)(1-\hat{\eta}^2)^{1/2}$$

The integrals in Eq. (46) then assume the form

$$\beta^{-1}\iint \frac{d\xi\, d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}}$$

The integrations with respect to ξ are carried out first. To determine the limits of integration, we determine the points of intersection of the straight lines $\hat{\eta}$=const in the $\xi\eta$-plane with the circle that forms part of the boundary of the region of integration. Substituting the definition of $\hat{\eta}$, Eq. (49), into the equation of the circle, Eq. (48), one obtains

$$((x-\xi)-Mt)^2 + (x-\xi)^2\hat{\eta}^2/(M^2-1) = t^2$$

This equation is first solved for $(x-\xi)$, but the result is written down for ξ.

Let $$B_1(\hat{\eta}) = \frac{tM(M^2-1)}{M^2-(1-\hat{\eta}^2)}, \quad B_2(\hat{\eta}) = \frac{t(M^2-1)(1-\hat{\eta}^2)^{1/2}}{M^2-(1-\hat{\eta}^2)} \tag{50}$$

Then the two roots for ξ are

$$\xi_1 = x - (B_1(\hat{\eta}) + B_2(\hat{\eta}))$$

$$\xi_2 = x - (B_1(\hat{\eta}) - B_2(\hat{\eta}))$$

These are the lower limits for ξ, if the line $\hat{\eta}$=const ends on the circle. If the line ξ=const ends at the leading edge, ξ=0.

Thus the lower limit is given by

in the region R_1 by $\max(0,\xi_1(\hat{\eta}))$

and in the region R_2 by $\max(0,\xi_2(\hat{\eta}))$

The upper limit is always $\xi=x$

Then one obtains along a line $\hat{\eta}$=const

$$\int d\xi = x - \max(0,\xi_1(\hat{\eta})$$

$$= \min\,(x,B_1(\hat{\eta}) + B_2(\hat{\eta}))\text{ in the region }R_1$$

$$\int d\xi = \min\,(x,B_1(\hat{\eta})-B_2(\hat{\eta}))\text{ in the region }R_2$$

Thus

$$\phi^{(s)}(t,x) = -\beta^{-1}\Big[\int_{\hat{\eta}=-1}^{\hat{\eta}=1}\frac{\min(x,B_1(\hat{\eta})+B_2(\hat{\eta}))}{(1-\hat{\eta}^2)^{1/2}}d\hat{\eta} + \int_{\hat{\eta}=-1}^{\hat{\eta}=+1}\frac{\min(x,B_1(\hat{\eta})-B_2(\hat{\eta})}{(1-\hat{\eta}^2)^{1/2}}d\hat{\eta}\Big] \tag{51}$$

The transition from one form of the minima to the other occurs at the points of intersection of the circle

$$(x-\xi-Mt)^2 + \eta^2 = t^2$$

with the leading edge, i.e., the line $\xi=0$. Hence, characterizing these points by a subscript 0

$$\eta_o = \pm [t^2 - (x-Mt)^2]^{1/2}$$

Now, for $\xi=0$

$$\hat{\eta}_0 = \beta \eta_o / x$$

Then

$$\hat{\eta}_0 = \pm\beta((\frac{t}{x}(M+1)-1)(1-\frac{t}{x}(M-1)))^{1/2}$$

or $$\hat{\eta}_0 = \pm\beta^2[(\frac{t}{x} - \frac{1}{M+1})(\frac{1}{M-1} - \frac{t}{x})]^{1/2} \qquad (52)$$

The transition from one form of the minima to the other then occurs for

$$\hat{\eta} = \hat{\eta}_0$$

The form of $\hat{\eta}_0$ shows that the points of intersection are real only if

$$(M+1)^{-1} < t/x < (M-1)^{-1}$$

Now specific expressions for the minima can be listed.

Case a: $t/x < (M+1)^{-1}$

for $-1 < \hat{\eta} < 1$, $\min(x, B_1+B_2) = B_1 + B_2$

$$\min(x, B_1-B_2) = B_1 - B_2$$

Case b: $(M+1)^{-1} < t/x < M(M^2-1)^{-1}$

for $-|\hat{\eta}_0| < \hat{\eta} < |\hat{\eta}_0|$, $\min(x, B_1+B_2) = x$

for $-1 < \hat{\eta} < -|\hat{\eta}_0|$

and $|\hat{\eta}_0| < \hat{\eta} < 1$, $\min(x, B_1+B_2) = B_1 + B_2$

for $-1 < \hat{\eta} < +1$, $\min(x, B_1-B_2) = B_1 - B_2$

Case c: $M(M^2-1)^{-1} < t/x < (M-1)^{-1}$

for $-1 < \hat{\eta} < 1$ $\min(x, B_1+B_2) = x$

for $-1 < \hat{\eta} < -|\hat{\eta}_0|$

and $|\hat{\eta}_0| < \hat{\eta} < 1$ $\min(x, B_1-B_2) = x$

for $-|\hat{\eta}_0| < \hat{\eta} < |\hat{\eta}_0|$ $\min(x, B_1-B_2) = B_1 - B_2$

Case d: $(M-1)^{-1} < t/x$

for $-1 < \hat{\eta} < 1$ $\min(x, B_1+B_2) = x$

$\min(x, B_1-B_2) = x$

These expressions are now substituted into Eq. (51), the resulting integrals are combined in such a manner that the limits are either $\hat{\eta}=-1$ and $\hat{\eta}=+1$, or $\hat{\eta}=-\hat{\eta}_0$ and $\hat{\eta}=+\hat{\eta}_0$. One obtains

Case a, $t/x < (M+1)^{-1}$

$$\phi(t,x) = -\beta^{-1}\left\{ \int_{-1}^{+1} \frac{(B_1+B_2)d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} + \int_{-1}^{+1} \frac{(B_1-B_2)d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} \right\}$$

$$\phi(t,x) = -2\beta^{-1} \int_{-1}^{+1} \frac{B_1 d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}}$$

Case b, $(M+1)^{-1} < t/x < M(M^2-1)-1$

$$\phi(t,x) = -\beta^{-1}\left\{ \int_{-1}^{+1} \frac{(B_1+B_2)d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} + \int_{-\hat{\eta}_0}^{+\hat{\eta}_0} \frac{x-(B_1+B_2)d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} + \int_{-1}^{+1} \frac{(B_1-B_2)d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}}\right\}$$

$$\phi(t,x) = -\beta^{-1}\left\{2 \int_{-1}^{+1} \frac{B_1 d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} + \int_{-\hat{\eta}_0}^{+\hat{\eta}_0} \frac{x-(B_1+B_2)d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}}\right.$$

Case c, $M(M^2-1)^{-1} < t/x < (M-1)^{-1}$

$$\phi(t,x) = -\beta^{-1}\left\{\int_{-1}^{+1} \frac{x d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}}d\hat{\eta} + \int_{-1}^{+1} \frac{x d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}}d\hat{\eta} - \int_{-\hat{\eta}_0}^{+\hat{\eta}_0} \frac{x d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} + \int_{-\hat{\eta}_0}^{+\hat{\eta}_0} \frac{(B_1-B_2)d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}}\right\}$$

$$\phi(t,x) = -\beta^{-1}\left\{2x \int_{-1}^{+1} \frac{d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} + \int_{-\hat{\eta}_0}^{+\hat{\eta}_0} \frac{(-x+B_1-B_2)d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}}\right\}$$

Case d, $(M-1)^{-1} < t/x$

$$\phi(t,x) = -2\beta^{-1} x \int_{-1}^{+1} \frac{d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}}$$

The integrals occurring in these expressions can be expressed by elementary transcendental functions

$$\int \frac{d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} = \arcsin\hat{\eta}$$

One obtains with Eq. (50)

$$\int \frac{B_2 d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} = t(M^2-1) \int \frac{d\hat{\eta}}{M^2-1+\hat{\eta}^2} = t(M^2-1)^{1/2} \text{arctg} \frac{\hat{\eta}}{(M^2-1)^{1/2}}$$

$$\int \frac{B_2 d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} = t\beta \, \text{arctg} \, \frac{\hat{\eta}}{\beta}$$

$$\int \frac{B_1 d\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}} = tM(M^2-1) \int \frac{d\hat{\eta}}{(M^2-1+\hat{\eta}^2)(1-\hat{\eta}^2)^{1/2}} = t\beta \, \text{arctg} \left(\frac{M}{\beta} \frac{\hat{\eta}}{(1-\hat{\eta}^2)^{1/2}}\right)$$

These formulae can be verified by differentiation. Regarding the last expression, it is shown in Appendix B how it can be derived from an integral found in Ref. 3.

One then obtains the following specific formulae

for $\quad t/x < (M+1)^{-1}$

$$\phi(x,t) = -2t\pi = -2x\pi(t/x)$$

for $\quad (M+1)^{-1} < (t/x) < M(M^2-1)^{-1}$

$$\phi(x,t) = -2x\Big[\pi(t/x) + \beta^{-1} \text{arc sin } \hat{\eta}_0$$

$$- (t/x) \text{ arctg } \left(\frac{M}{\beta} \frac{\hat{\eta}_0}{(1-\hat{\eta}_0^2)^{1/2}}\right) - (t/x) \text{arctg } \frac{\hat{\eta}_0}{\beta}\Big]$$

for $\quad M(M^2-1) < t/x < (M-1)^{-1}$

$$\phi(t,x) = -2x\Big[\beta^{-1}\pi - \beta^{-1} \text{arc sin } \hat{\eta}_0 + (t/x) \text{ arctg } \left(\frac{M}{\beta} \frac{\hat{\eta}_0}{(1-\hat{\eta}_0{}^2)^{1/2}}\right)$$

$$- (t/x) \text{ arctg } \left(\frac{\hat{\eta}_0}{\beta}\right)\Big]$$

for $(M^2-1)^{-1} < t/x$

$$\phi(t,x) = -2x\beta^{-1}\pi .$$

$\hat{\eta}_0$ is found in Eq. (52).

It is probably best to evaluate the expressions for fixed t, as a function of x.

APPENDIX A

EVALUATION OF SOME INTEGRALS

The integral

$$I = \int_{-1}^{+1} \frac{d\xi(1-\xi^2)^{1/2}}{(\xi-M)^2}$$

can be decomposed into

$$I = -\int_{-1}^{+1} \left(1 + \frac{2M}{\xi-M} + \frac{M^2-1}{(\xi-M)^2}\right) \frac{d\xi}{(1-\xi^2)^{1/2}}$$

One has

$$\int_{-1}^{+1} \frac{d\xi}{(1-\xi^2)^{1/2}} = \text{arc sin}\xi \Big|_{-1}^{+1} = \pi$$

The other integrals are found in tables, for instance Ref. 3, Formula 236, 3c

$$\int \frac{dx}{(x-\alpha)(a^2-x^2)^{1/2}} = \frac{1}{(\alpha^2-a^2)^{1/2}} \text{ arc sin } \frac{a^2-\alpha x}{a|x-\alpha|} \text{ , } |\alpha| > a$$

Therefore

$$\int_{-1}^{+1} \frac{d\xi}{(\xi-M)(1-\xi^2)^{1/2}}$$

$$= \frac{1}{(M^2-1)^{1/2}} \text{arc sin } \frac{1-M\xi}{M-\xi} \Big|_{\xi=-1}^{\xi=+1}$$

$$= \frac{1}{(M^2-1)^{1/2}} [\text{arc sin}\frac{1-M}{M-1} - \text{arc sin } \frac{1+M}{1+M}] = -\frac{\pi}{(M^2-1)^{1/2}}$$

Formula 236, 3a of Ref. 3 reads

$$\int \frac{d\xi}{(x-\alpha)^k (a^2-x^2)^{1/2}} = \frac{-1}{(k-1)(a^2-\alpha^2)} \left[\frac{(a^2-x^2)^{1/2}}{(x-\alpha)^{k-1}} - (2k-3)\alpha \int \frac{dx}{(x-\alpha)^{k-1}(a^2-x^2)^{1/2}} \right.$$

$$\left. - (k-2) \int \frac{dx}{(x-\alpha)^{k-2}(a^2-x^2)^{1/2}} \right]$$

Hence for k=2

$$\int_{-1}^{+1} \frac{d\xi}{(\xi-M)^2(1-\xi^2)^{1/2}} = -\frac{1}{1-M^2} \left[\frac{(1-\xi^2)^{1/2}}{(\xi-M)} - M \int \frac{d\xi}{(\xi-M)(1-\xi^2)^{1/2}} \right] \Bigg|_{-1}^{+1}$$

$$= \frac{M\pi}{(M^2-1)^{3/2}}$$

Thus

$$\int_{-1}^{+1} \frac{d\xi(1-\xi^2)^{1/2}}{(\xi-M)^2} = \pi\left(-1 + \frac{2M}{(M^2-1)^{1/2}} - \frac{M}{(M^2-1)^{1/2}}\right) = \pi\left(-1 + \frac{M}{\beta}\right)$$

In the next integral, a decomposition is carried out, so that the first order poles are displayed.

$$\int \frac{d\eta}{(\eta^2+\beta^2)(1-\eta^2)^{1/2}} = \frac{1}{2i\beta} \int \left[\frac{1}{(\eta-i\beta)(1-\eta^2)^{1/2}} - \frac{1}{(\eta+i\beta)(1-\eta^2)^{1/2}} \right] d\eta$$

Therefore

$$\int \frac{d\eta}{(\eta^2+\beta^2)(1-\eta^2)^{1/2}} = \frac{1}{\beta} \operatorname{Im} \int \frac{d\eta}{(\eta-i\beta)(1-\eta^2)^{1/2}} \qquad (A.1)$$

Now one finds in Ref. 3, formula 236, 3c

$$\int \frac{dx}{(x-\alpha)(a^2-x^2)^{1/2}} = \frac{-1}{(a^2-\alpha^2)^{1/2}} \log \frac{-\alpha x + a^2 + [(a^2-\alpha^2)(a^2-x^2)]^{1/2}}{x-\alpha}$$

Specializing for $a=1$, $\alpha=i\beta$, $x=\eta$, one obtains, with $\beta^2=M^2-1$,

$$\int \frac{d\eta}{(\eta - i\beta)(1-\eta^2)^{1/2}} = -\frac{1}{M} \log \left(\frac{-i\beta\eta + 1 + M(1-\eta^2)^{1/2}}{\eta - i\beta}\right)$$

$$= -\frac{1}{M} \log \frac{(-i\beta\eta + 1 + M(1-\eta^2)^{1/2})(\eta + i\beta)}{\eta^2 + \beta^2}$$

$$= -\frac{1}{M} \log \frac{M\eta(M + (1-\eta^2)^{1/2} + i\beta(1-\eta^2)^{1/2}(M + (1-\eta^2)^{1/2})}{\eta^2 + \beta^2}$$

$$= -\frac{1}{M}\left[\log\left[\frac{M + (1-\eta^2)^{1/2}}{\eta^2 + \beta^2}(\beta(1-\eta^2)^{1/2} - M\eta)\right] + \log i\right]$$

The last term in the bracket is a constant of integration which can be disregarded.

Therefore, from Eq. (A.1)

$$\int \frac{d\eta}{(\eta^2 + \beta^2)(1-\eta^2)^{1/2}} = \frac{1}{M\beta} \text{ arc tg } \frac{M\eta}{\beta(1-\eta^2)^{1/2}}$$

APPENDIX B

REGIONS OF INTEGRATION FOR $z \neq 0$

Before he discussed the one-dimensional time dependent flow for $z=0$ (Section IV), the author investigated this problem for $z \neq 0$. One obtains, as expected, the results usually derived from the partial differential equation. In this appendix an intermediate result is shown, namely how, for fixed x,y,z, the regions of integration change with time. In Section IV the corresponding regions have been shown for $z=0$. It is of interest to observe the relation between these cases.

It is assumed that the time dependence of the upwash is given by a Hamilton step-function. The arguments are given by

$$\arg_{1,2} = t - \text{ret} = t - \beta^{-2}(M(x-\xi) \pm R) \tag{B.1}$$

with

$$R = [(x-\xi)^2 - \beta^2(\eta-y)^2 - \beta^2 z^2]^{1/2} > 0.$$

The areas of integration at fixed x,y,z,t are characterized by the conditions, that R is real, and that $\arg_1 > 0$, $\arg_2 > 0$. In the present special example, all points of the $\xi\eta$-plane are equivalent, there is no leading edge. In the general case the region of integration has the same boundaries, except that points (ξ,η) upstream of the leading edge are excluded.

Part of the boundary of the region of integration is given in the $\xi\eta$-plane by the hyperbola $R=0$

$$(x-\xi)^2 - \beta^2(\eta-y)^2 - \beta^2 z^2 = 0 \tag{B.2}$$

In the interior of the hyperbola $R>0$. Only the branch for which $x-\xi > 0$ is of interest.

The vertex of this hyperbola lies at

$$\xi = x-\beta z; \; z > 0)$$

$$\eta = y$$

Its asymptotes are given by

$$\eta - y = \pm \beta^{-1} (x-\xi)$$

Other portions of the boundary of the region of integration are given by the curve along which $\arg_1$ or $\arg_2$ vanish. Substituting β^2 one obtains from Eq. (B.1)

$$(M^2-1)t - M(x-\xi) = \pm \left[(x-\xi)^2 - (M^2-1)(\eta-y)^2 - (M^2-1)z^2 \right]^{1/2}$$

Squaring both sides to remove the square root and dividing by M^2-1, one obtains

$$M^2t^2 - t^2 - 2M(x-\xi)t + (x-\xi)^2 + (\eta-y)^2 + z^2=0$$

or

$$(\xi-(x-Mt))^2 + (\eta-y)^2 = t^2-z^2 \qquad \text{(B.3)}$$

This is a circle with center at $\xi=x-Mt$, $\eta=y$ and radius $(t^2-z^2)^{1/2}$. One surmises that the hyperbola R=0 is tangent to this circle. Elimination of $(\eta-y)$, from Eqs. (B.2 and B.3) gives the values of ξ (or rather $(x-\xi)$) for the points of intersection of the two curves. Actually, this process eliminates z at the same time. One obtains

$$(M(x-\xi) - (M^2-1)t)^2 = 0$$

The fact that one obtains a double root for $x-\xi$ indicates that the curves given by Eqs. (B.2 and B.3) are tangent to each other

(if they have points in common at all). The points of tangency lie at

$$\xi = x - \frac{M^2-1}{M}.$$

The corresponding values of η are found from Eq. (B.3)

$$(\eta-y)^2 = \pm \frac{M^2-1}{M^2} t^2 - z^2$$

The points of tangency divide the circle for which $\arg_1 = 0$ and $\arg_2 = 0$ into two parts. At the point of tangency as along the entire hyperbola, R=0. In the interior of the hyperbola $R > 0$. If one travels along the circle starting at the point of tangency in the direction of decreasing $(x-\xi)$, then $\arg_1$ (upper sign of R) is zero, and $\arg_2$ becomes positive; this portion of the contour is therefore the boundary of the region there $\arg_1 > 0$. The portion of the circle for which $x-\xi$ is greater than the value for the point of tangency is the boundary of the region for which $\arg_2 > 0$.

The smallest value of t, for which points of tangency exist, arises if the circle is tangent to the hyperbola at its vertex. Then $(\eta-y) = 0$ and

$$t = M(M^2-1)^{-1/2}z.$$

This, however, is not the smallest possible of these circles. The radius is given by (t^2-z^2). The smallest circle therefore arises for t=z. This is a point in the interior of the hyperbola. There $R > 0$. These circles in their entirety are boundaries of regions where $\arg_2 > 0$. One thus obtains the sequence of boundaries shown in Figure 5. Those boundaries in Figure 5b that consist of full circles are not present if z=0.

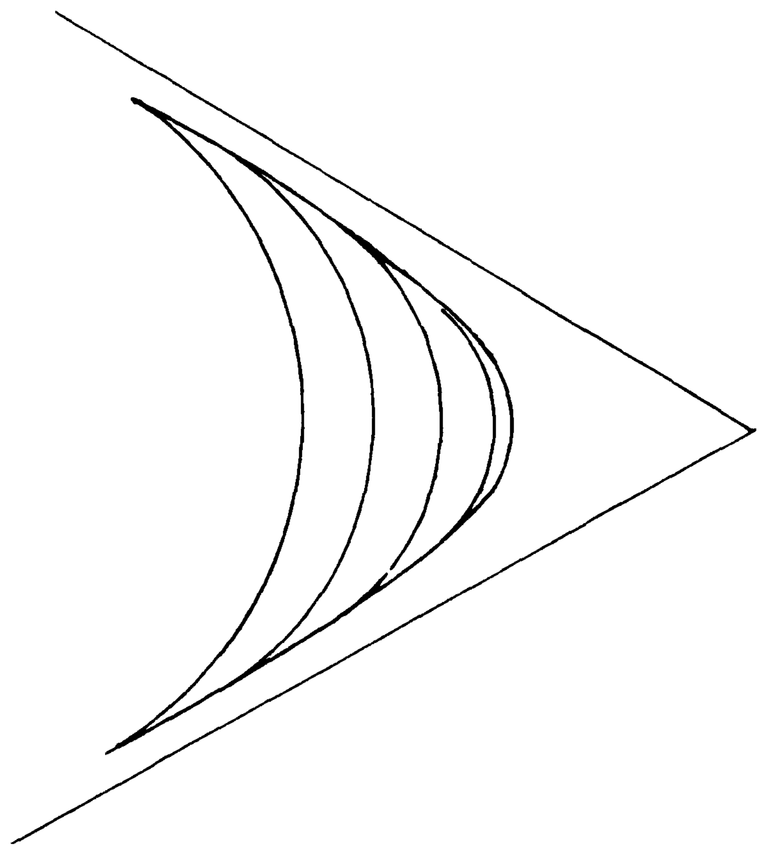


Figure 5a. Regions of Integration for $z \neq 0$ at Different Times. The Boundaries are Given by Parts of Circle and by Parts of an Enveloping Hyperbola.

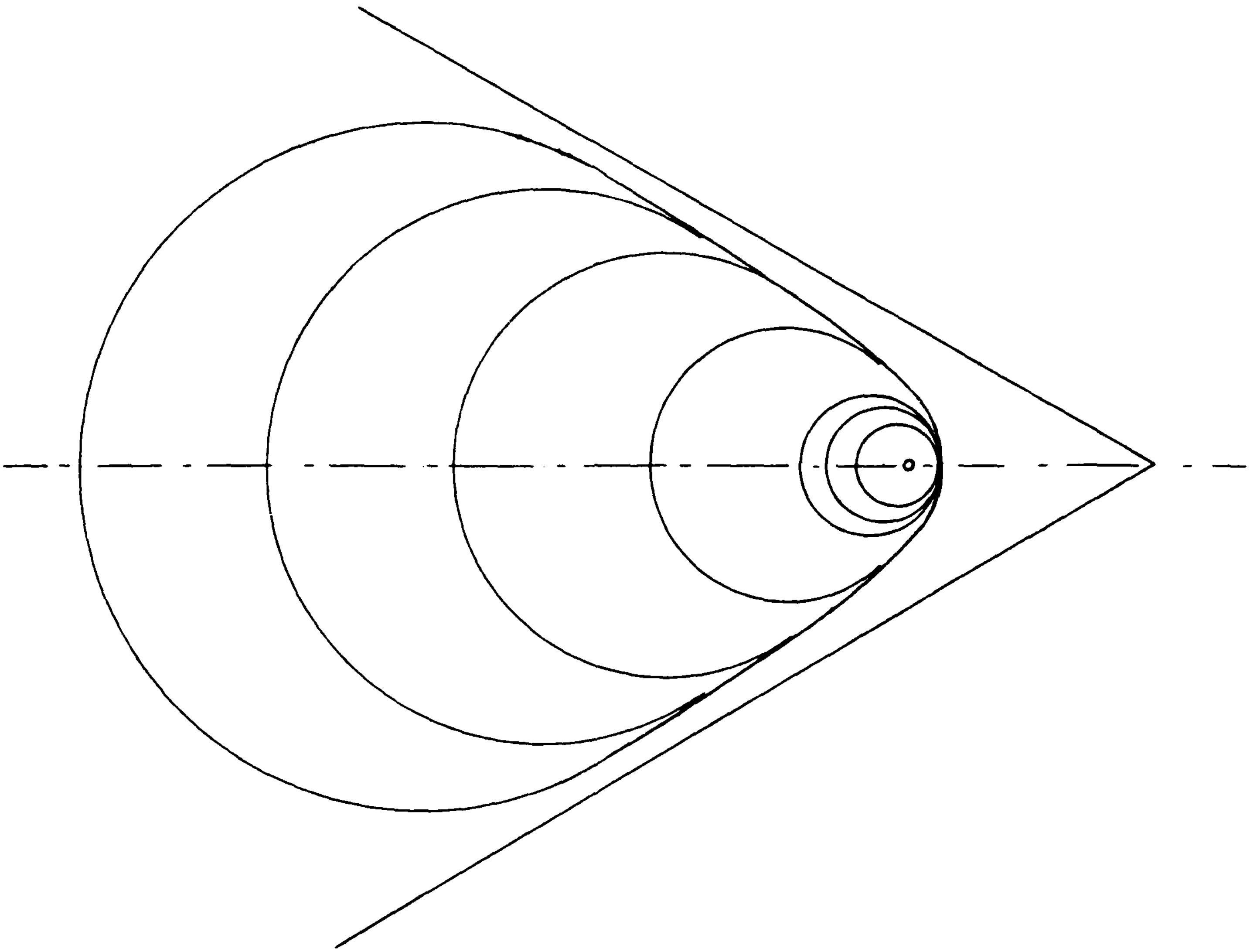


Figure 5b. Regions of Integration for $z \neq 0$ at Various Times. For Small Time the Boundaries are Circles. For Greater Times each Boundary consists of Part of a Circle and Part of a Hyperbola.

APPENDIX C

DISCUSSIONS RELATED TO FUNDAMENTAL SOLUTIONS

Fundamental solutions can, of course, be derived directly from the partial differential equation for the perturbation potential in a system of coordinates in which the air is in motion. Here one has

$$-(M^2-1)\phi_{xx} + \phi_{yy} + \phi_{zz} - 2M\phi_{xt} - \phi_{tt} = 0 \qquad (C.1)$$

Let

$$\hat{x} = x, \quad \hat{y} = y(M^2-1)^{1/2}, \quad \hat{z} = z(M^2-1)^{1/2} \qquad (C.2)$$

and consider perturbations which are harmonic in time

$$\phi = \hat{\phi}(\hat{x},\hat{y},\hat{z}) \exp(i\nu t) \qquad (C.3)$$

Then

$$-\hat{\phi}_{\hat{x}\hat{x}} + \hat{\phi}_{\hat{y}\hat{y}} + \hat{\phi}_{\hat{z}\hat{z}} - \frac{2i\nu M}{M^2-1}\hat{\phi}_{\hat{x}} + \frac{\nu^2}{M^2-1}\hat{\phi} = 0 \qquad (C.4)$$

The derivative $\hat{\phi}_{\hat{x}}$ is suppressed if one introduces

$$\hat{\phi} = \tilde{\phi} \exp\left(\frac{-i\nu M}{M^2-1}\hat{x}\right) \qquad (C.5)$$

One obtains

$$-\tilde{\phi}_{\hat{x}\hat{x}} + \tilde{\phi}_{\hat{y}\hat{y}} + \tilde{\phi}_{\hat{z}\hat{z}} - \frac{\nu^2}{(M^2-1)^2}\tilde{\phi} = 0 \qquad (C.6)$$

Particular solutions are obtained as follows. Let

$$R^2 = \hat{x}^2 - (\hat{y}^2 + \hat{z}^2)$$

and assume that $\hat{\phi}$ depends upon R only; then

$$\tilde{\phi}_{RR} + \frac{2}{R}\tilde{\phi}_R + \frac{\nu^2}{M^2-1}\tilde{\phi} = 0$$

This is solved by

$$\tilde{\phi} = (1/R)\ \exp\left(\frac{\pm i\nu}{M^2-1}\right)R \qquad (C.7)$$

Returning to the original coordinate one has

$$R^2 = x^2 - (M^2-1)(y^2+z^2)$$

$$\phi = (1/R)\ \exp\left(i\nu t - \frac{i\nu Mx}{M^2-1} \pm \frac{i\nu}{M^2-1}\ R\right)$$

These are, of course, the particular solutions encountered in Eq. (20). But the present derivation does not show, which linear combination should be taken. Besides, it is not clear, whether the continuation of these particular solutions outside of the cone $R > 0$, is given by $\phi = 0$. We shall see, that this is not always the case.

Eq. (C.6) has no physical significance because of the splitting off of the factor $\exp\left(\frac{-i\nu M}{M^2-1}\ \hat{x}\right)$ in Eq. (C.5). Since this factor depends upon ν this procedure has no counterpart in the general nonsteady problem. The following observations may, nevertheless, be of interest.

Let

$$\frac{\nu}{M^2-1} = \tilde{\nu}$$

Then one has

$$\tilde{\phi}_{\hat{x}\hat{x}} - \tilde{\phi}_{\hat{y}\hat{y}} + \tilde{\phi}_{\hat{z}\hat{z}} + \tilde{\nu}^2\tilde{\phi} = 0 \qquad (C.8)$$

It is remarkable that in this equation $\tilde{\nu}$ occurs in the second power, while in the solution Eq. (C.7) it appears in the first power. If, however, one separates in Eq. (C.7) real and imaginary part one has

$$\frac{1}{R}\cos(\tilde{\nu}R), \text{ and } \frac{1}{R}\sin(\tilde{\nu}R) \qquad (C.9)$$

and the development of these expressions proceed in powers of $\tilde{\nu}^2$. One can think of solving Eq. (C.8) by a development with respect to $\tilde{\nu}^2$. To the lowest order one then has

$$\tilde{\phi}_{RR} + \frac{2}{R}\tilde{\phi}_R = 0$$

with the solutions

$$\tilde{\phi} = (1/R) \text{ and } \tilde{\phi} = 1$$

$\phi=(1/R)$ is the first term of the development of $(1/R)\cos\tilde{\nu}R$, $\phi=1$ is the first term of the development of $(1/R)\sin\tilde{\nu}R$ (after multiplication by ν^{-1}).

There is a remarkable difference in these expressions which will be explored further in this appendix. The solution $\tilde{\phi} = 1/R$ is real only within the Mach cone. But it can be continued outside by $\tilde{\phi}=0$, and the expression satisfies the differential equation even as one passes through the Mach cone.

The expression $\phi=1$ holds everywhere. But because of the nature of supersonic flows one does not admit perturbations outside of the aftercone of the point xyz. Therefore, one is inclined to set $\phi=0$ outside of the aftercone. But then one fails to satisfy the flow differential equation as one passes through the cone.

We discuss the case $\tilde{\nu}=0$ further; it is the first step in a development with respect to $\tilde{\nu}$. The function ϕ is the perturbation potential, its gradient describes the perturbations in the velocity field. Let us discuss the perturbations in the

mass flow vector $\rho\vec{w}$ (where ρ is the density). Let ρ_o be the density of the basic flow

$$\Delta\rho\vec{w} = \rho\vec{w} - \rho_o U\vec{e}_x$$

Now

$$\vec{w} = U\vec{e}_x + \Delta\vec{w}$$

$$\Delta\vec{w} = \phi_x \vec{e}_x + \phi_y \vec{e}_y + \phi_z \vec{e}_z$$

and $$\rho = \rho_o + \Delta\rho$$

One has $$\frac{dp}{d\rho} = 1/a^2$$

The perturbation in the pressure Δp is found from Bernoulli's equation

$$\Delta p = -\rho_o U\phi_x$$

Then $$\Delta\rho = -\rho_o U\phi_x/a^2 = -\rho_o M\phi_x/a$$

Therefore

$$\rho\vec{w} = \rho_o(1-M\phi_x/a)(\vec{e}_x(U+\phi_x) + \vec{e}_y\phi_y + \vec{e}_z\phi_z)$$

$$\frac{\Delta\rho\vec{w}}{\rho 0} = \vec{e}_x\phi_x(1-M^2) + \vec{e}_y\phi_y + \vec{e}_z\phi_z$$

The potential equation expresses conservation of mass. If it is satisfied, then one has

$$\mathrm{div}(\rho\vec{w}) = 0$$

and for the linearized form of the potential equation

$$\mathrm{div}(\Delta\rho\vec{w}) = 0.$$

In the cylindrical coordinates x,y this equation assumes the form

$$\frac{\partial}{\partial x}(y\Delta(\rho\vec{w})_x + \frac{\partial}{\partial y}(y\Delta(\rho\vec{w})y) = 0$$

This equation is satisfied by introducing a "stream function" Δy for the perturbed flow

$$y\Delta(\rho\vec{w})_x = 2\pi\Delta\psi_y$$

$$y\Delta(\rho\vec{w})_y = -2\pi\Delta\psi_x$$

Then

$$\Delta\psi = 2\pi\int(y\Delta(\rho\vec{w})_x dy - y\Delta(\rho\vec{w})dx)$$

If one chooses two points A and B (Figure 6) and connects them by some curve and evaluates the last integral along this curve, then the result is independent of the choice of this curve. Therefore, $\Delta\psi$ is a function of x and y. The difference $\Delta\psi_B - \Delta\psi_A$ is the perturbation in the mass flow passing through the axisymmetric surface generated by rotating the curve AB with the meridian plane around the x axis. Since $\Delta\psi$ depends upon x and y only, one can draw "stream lines" of the undisturbed flow (lines $\Delta\psi$=const). The total mass flow through the surface AB is, of course, the mass flow due to the unperturbed flow plus the contribution of the perturbation.

This is applied to the fundamental solution (Figure 7)

$$\phi = (x^2-(M^2-1)y^2)^{-1/2} \quad , \; x>0, \; |y|<(M^2-1)^{-1/2}x$$

Then

$$\phi_x = -x(x^2-(M^2-1)y^2)^{-3/2}$$

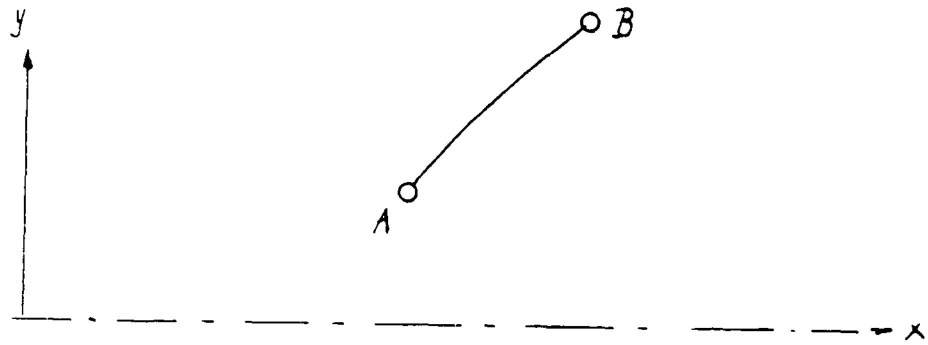


Figure 6. Curve for the Evaluation of $\Delta\psi_B - \Delta\psi_A$.

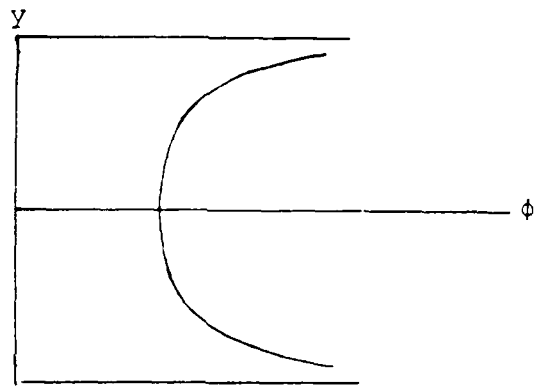


Figure 7. Potential at Fixed Value of x versus y in a Supersonic Source Flow. The Asymptote Corresponds to the Mach Cone.

$$\phi_y = (M^2-1)y(x^2-(M^2-1)y^2)^{-3/2}$$

$$\Delta(\rho\vec{w}) = (M^2-1)(\vec{e}_x x + \vec{e}_y y)(x^2-(M^2-1)y^2)^{-3/2}$$

This shows that the stream lines pertaining to the perturbation are radii y/x = const, although the pertinent velocity vectors have a different direction. For simplicity we set in the following discussions $M^2-1=1$. Let us determine the values of $\Delta\psi$ as functions of y/x We set $\Delta\psi=0$ for $y/x=0$. It suffices if we carry out the integration along a line x = const. Let $y/x = a_1$

$$\Delta\psi(a_1) = 2\pi \int_0^{a_1} \frac{xydy}{(x^2-y^2)^{3/2}} = 2\pi x(x^2-y^2)^{-1/2} \Big|_0^{a_1} = 2\pi((1-a_1^2)^{-1/2}-1)$$

As $a_1 \to 1$, $\Delta\psi(a_1) \to \infty$; the mass flow due to the perturbation within the Mach cone is infinite.

According to the derivations of Section I, the particular solution $\phi = (x^2-y^2)^{-1/2}$ is valid throughout the xy plane, for $y/x > 1$ it should be continued by $\phi=0$, and one must take into account the infinite jump at $y/x=1$.

To get some insight, we replace the jump by a narrow transition region $1-\varepsilon < y/x < 1$ (Figure 8). There we set

$$\phi = \frac{1}{x[1-(1-\varepsilon)^2]^{1/2}} \frac{1-y/x}{\varepsilon} = \frac{1}{(2-\varepsilon)^{1/2}\varepsilon^{3/2}}\left(\frac{1}{x} - \frac{y}{x^2}\right) \qquad \text{(C.10)}$$

At $y/x = 1-\varepsilon$, this matches the original expression

$$\phi = (x^2-y^2)^{-1/2}$$

At $y/x = 1$, it gives 0; i.e., it matches the outer field $y=0$. We compute the perturbation mass flow in the transition region:

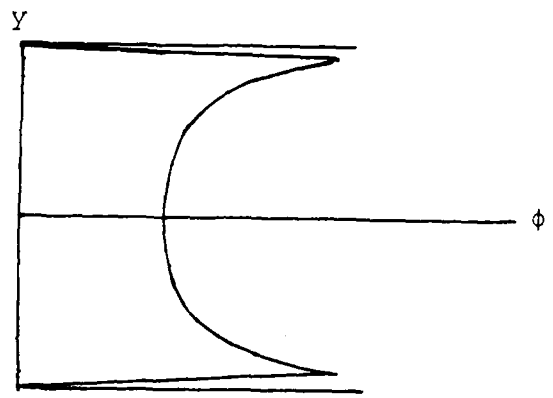


Figure 8. Modified Potential at a Fixed Value of x Versus y for a Supersonic Source. The jump at the Mach Cone is Replaced by a Transition in a Narrow Region.

$$-2\pi\varepsilon^{-3/2}(2-\varepsilon)^{-1/2}\int_{y=x(1-\varepsilon)}^{y=x}\left(-\frac{1}{x^2}+\frac{2y}{x^3}\right)y\,dy$$

$$=2\pi\varepsilon^{-3/2}(2-\varepsilon)^{-1/2}\left[((1/2)(y/x)^2-(2/3)(y/x)^3]\right|_{y/x=1-\varepsilon}^{y/x=1}$$

One obtains for ε small as mass flow in the transition region

$$-2\pi\varepsilon^{-1/2}2^{-1/2}$$

The perturbation mass flow between $y/x=0$ and $y/x=1-\varepsilon$ from the original expression ϕ is

$$2\pi\ [(2\varepsilon)^{-1/2}-1]$$

The total mass flow between $y/x=0$ and $y/x=1$ is therefore -2π. The physical picture for the perturbation mass flow (not for perturbation velocity) is a very concentrated inflow in the transition region and an outflow for $y/x < 1-\varepsilon$, which becomes very large as y/x approaches $(1-\varepsilon)$. The total inflow exceeds the outflow by π (Figure 9).

In the transition region the potential equation (the equation for conservation of mass) is not satisfied, in other words, one will find sources. This is seen in detail if one computes the perturbation mass flow through a conical surface $y=\alpha x$, $1-\varepsilon < \alpha > 1$, extending from x_1 to x_2. We choose $x_1 < x_2$ then the following expression is an outflow from the region $y/x > \alpha$, for one travels around the region in the positive sense. The potential is given by Eq. (C.10). One has

$$y = \alpha x$$

$$dy = \alpha dx$$

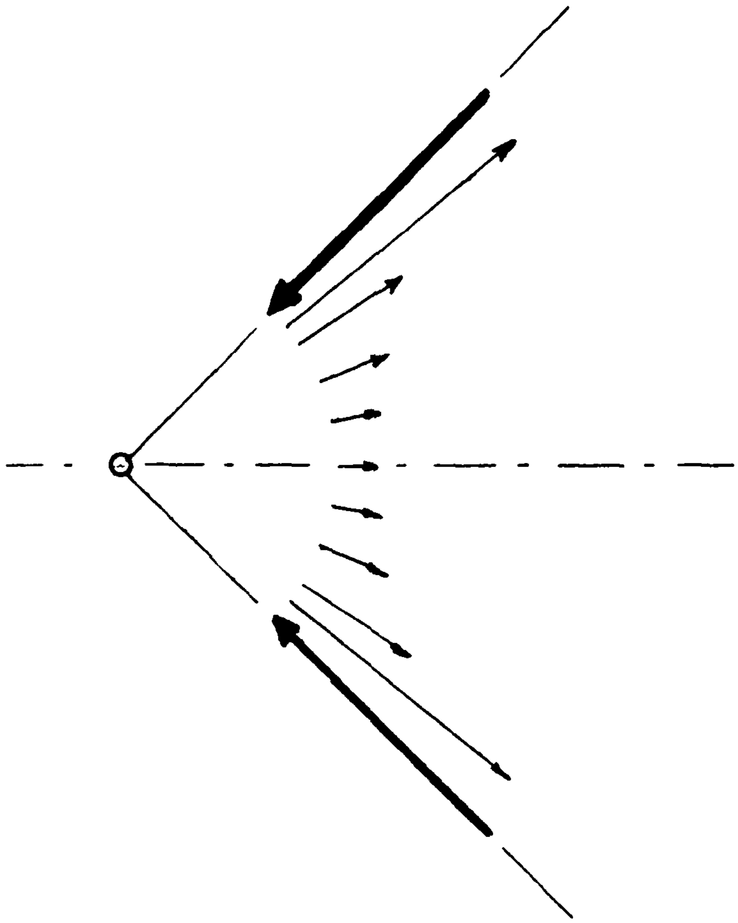


Figure 9. Perturbation Mass Flow Vector in Supersonic Source Flow at Some Radius within the Mach Cone. The Perturbation Mass Flow is Finite in Radial Direction; the Vector tends strongly to Infinity as one Approaches the Surface of the Mach Cone. The Total Mass Flow within the Mach Cone is Infinite. Concentrated Flow Back at the Surface of the Mach Cone, exceeding the Total Outflow by zτ. Therefore the Total Configuration Amounts to a Source with Negative Strength.

The perturbation mass flow through this section for $M^2-1 = 1$ is then given by

$$2\pi \int_{x_1}^{x_2} (-\phi_x dy - \phi_y dx) = -2\pi\alpha \int_{x_1}^{x_2} (-\alpha\phi x + \phi y) x dx$$

$$2\pi\alpha\varepsilon^{-3/2} 2^{-1/2} \int_{x_1}^{x_2} \left(\alpha\left(\frac{1}{x^2} - \frac{2\alpha}{x^2}\right) + \frac{1}{x^2}\right) x dx$$

$$= 2\pi\alpha(1-\alpha)(1+2\alpha)\varepsilon^{-3/2} 2^{-1/2} \log(x_2/x_y)$$

For α close to one, one obtains

$$6.2^{-1/2}\ \varepsilon^{-3/2}\ \pi \log\ (x_2/x_1)(1-\alpha) \tag{C.11}$$

There is no flow through the line $\alpha=1$. In the transition region $1-\alpha < \varepsilon$. The last expression is therefore $\log(x_2/x_1)\ O(\varepsilon^{-1/2})$. The expression (C.11) is the outflow from the region $x_1 < x < x_2$, $\alpha < y/x < 1$, since there is no net flow through the combined cross sections $x=x_1$ and $x=x_2$. The expression therefore represents the total of the sources within the region just described. This total behaves as $\varepsilon^{-1/2}$ as $z \to 0$.

There is an outflow as one approaches the line $y/x=1-\varepsilon$ from above, and no outflow as one approaches it from below. At the line $y/x=1-\varepsilon$ one therefore has concentrated sinks, which swallow the total of the sources from the region $\alpha < y/x) < 1$.

The sources in the interior of the region and the sinks along the boundary $y/x=1-\varepsilon$ can be combined to form doublets. Since the distance between the sources and sinks is $O(\varepsilon)$ the total doublet strength of the transition region is $O(\varepsilon^{1/2})$. This is the interpretation for the source solution.

If one tries to use a similar interpretation for a potential

$$\phi = 1, \quad y/x < 1$$

$$\phi = 0 \quad y/x > 1$$

which is the first term of a development of $\frac{1}{R} \sin \nu R$, one finds that the flow in the transition region through cross sections x = const is proportional to x_2. The total source strength in the transition region is obviously not zero, these solutions are therefore not admissible. At least for flows that are periodic in time this shows, that only the first of the expressions shown in Eq. (C.9) can be used to form a fundamental solution.

REFERENCES

1. Karl G. Guderley, Maxwell Blair, The Integral Equation for the Time Dependent Linearized Potential Flow Over a Wing. AFWAL-TR-86-3077, Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Air Force Systems Command, Wright-Patterson AFB, Ohio 45433-6553.

2. Karl G. Guderley, A Discretization of the Integral Equation for the Time Dependent Linearized Potential Flow Over a Wing. AFWAL-TR-87-3061, Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Air Force Systems Command, Wright-Patterson AFB, Ohio 45433-6553.

3. W. Gröbner, M. Hofreiter, N. Hofreiter, J. Laub, E. Peschl, Integraltafel, I. Teil. Unbestimmte Integrale. Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des GeneralLuftzeugmeisters - Berlin Adlerhof (has also appeared as a book).

4. Karl. G. Guderley, An Integral Equation for the Linearized Supersonic Flow Over a Wing. AFWAL-TR-87-3107, Flight Dynamics Laboratory.

SUMMARY

The report derives an integral equation for the linearized supersonic unsteady potential flow over a wing. Every integral equation formulation for a problem that appears originally in the form of a partial differential equation presupposes the availability of a fundamental solution. Such a fundamental solution is available for the problem at hand in the literature. It is rederived here to show its particular properties; further discussions are found in Appendix C. The integral equation originally obtained requires that one carry out a limiting process in which one approaches the planform from above or below. This formulation is brought into a form in which this limiting process no longer appears and one works solely with information available at the planform. Examples which can be treated analytically bring some properties which have a bearing on a numerical approach into sharper focus.

☆ U.S. GOVERNMENT PRINTING OFFICE 1988 548-054-807 0 6

END
DATE
FILMED
8-88
DTIC